4.1 Relativistic description of particles

A relativistic particle is characterized by:

- 4-vector of position:

$$\left(ct, \vec{x}\right) \equiv \left(x^0, x^1, x^2, x^3\right) \equiv x^\mu \equiv x \qquad (1)$$

- 4-momentum:

$$\left(E/c, \vec{x}\right) \equiv \left(p^0, p^1, p^2, p^3\right) \equiv p^\mu \equiv p \qquad (2)$$

4-vectors x^μ and p^μ are elements of 4-dimensional pseudo-Euclidian space M_4 (space *coordinates and of momenta separately*).

⇒ Properties of M_4

- Scalar product of two 4-vectors (in M_4) $A^\mu = \left(A^0, \vec{A}\right)$ a $B^\mu = \left(B^0, \vec{B}\right)$:

$$A \cdot B = A^0 B^0 - \vec{A} \cdot \vec{B} \qquad (3)$$

- Rotations in $M_4 \equiv$ Lorentz transformation ⇒ they create the Lorentz group (LG).

Scalar product is an invariant of LG

- Covariant 4-vector: $A^\mu = \left(A^0, -\vec{A}\right) \rightarrow$ for scalar product is valid:

$$A \cdot B = A^\mu B_\mu = A_\mu B^\mu = g_{\mu\nu} A^\mu B^\nu = g^{\mu\nu} A_\mu B_\nu \qquad (4)$$

where $g_{\mu\nu}$ is the metric tensor:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (5)$$

- Invariants of LG

→ Interval: $$x_\mu x^\mu = c^2 t^2 - \vec{x}^2 \qquad (6a)$$

→ Square of mass: $$p_\mu p^\mu = \frac{E^2}{c^2} - \vec{p}^2 = m^2 c^2 \qquad (6b)$$

→ Phase: $$p_\mu x^\mu = Et - \vec{p} \cdot \vec{x} \qquad (6c)$$

The natural units: $c = 1$ a $\hbar = 1$ (7)

$\rightarrow \quad E^2 - \vec{p}^2 = m^2$

$\rightarrow$ QM – assignment of operator:

$$\vec{p} \rightarrow -i\hbar\nabla \equiv -i\hbar\left(\frac{\partial}{\partial x^1}\ ,\ \frac{\partial}{\partial x^2}\ ,\ \frac{\partial}{\partial x^3}\right) \quad \text{a} \quad E \rightarrow i\hbar\frac{\partial}{\partial t} \tag{8a}$$

In the natural units:

$$(E, \vec{p}) \rightarrow i\left(\frac{\partial}{\partial t}\ , -\nabla\right) \equiv i\partial^{\mu} \tag{8b}$$

Covariant form of operator ∂ : $\partial_{\mu} \equiv \left(\frac{\partial}{\partial t}, \nabla\right)$ (8c)

The shorten writings:

$$\begin{array}{ccc} \frac{\partial}{\partial t} \equiv & \partial_t \equiv & \partial_0 \\ . & . & . \\ \frac{\partial}{\partial x^3} \equiv & \partial_z & \partial_3 \end{array} \tag{9}$$

4.2 Klein - Gordon equation. Antiparticle

Let us have a relativistic particle with momentum $\vec{p}$, the relation for energy is:

$E^2 = \vec{p}^2 + m^2$. If we substitute E and $\vec{p}$ by operators as in QM:

$$(E, \vec{p}) \rightarrow i(\partial_t, \nabla),$$

From the relation for energy we obtain the Klein-Gordon(KG) equation:

$$-\frac{\partial^2}{\partial t^2}\varphi + \nabla^2\varphi = m^2\varphi \quad \textbf{resp.} \quad (\partial_{\mu}\partial^{\mu} + m^2)\varphi = 0 \tag{2.1}$$

Where $\partial_{\mu}\partial^{\mu} = \partial_t^2 - \nabla^2$ is D´Alembert’s operator a $\varphi(t, \vec{r})$ is a complex function from which we expect that the square of its module $|\varphi|^2$ will be a probability density to find particle in the position $\vec{r}$.

Solution for a free particle with momentum $\vec{p}$:

$$\varphi(t,\vec{x}) = N \cdot e^{-i(Et-\vec{p}\vec{x})} = N \cdot e^{-ipx} \qquad (2.2)$$

Where

$E = \pm\sqrt{\vec{p}^2 + m^2}$ are eigenvalues of energy,

N is the normalization constant: $|\varphi|^2 = |N|^2$ is the density of particles.

Problem:

What does it mean the solution $E = -\sqrt{\vec{p}^2 + m^2} < 0$? This solution we cannot throw away as the system of states would not be full.

Continuity equation (CE).

From KG equation using the following re-arrangement: $-i\varphi^* \cdot KG + i\varphi \cdot KG^*$

We obtain:

$$\partial_t \underbrace{\left[i\left(\varphi^*\partial_t\varphi - \varphi\partial_t\varphi^*\right)\right]}_{\rho = j_0} + \nabla \cdot \underbrace{\left[i\left(\varphi^*\nabla\varphi - \varphi\nabla\varphi^*\right)\right]}_{\vec{j}} = 0 \qquad (2.3a)$$

hence

$$\partial_t \rho + \nabla \cdot \vec{j} \equiv \partial_\mu j^\mu = 0 \qquad (2.3b)$$

Where j^μ is 4-vektor of current density connected with the solution $\varphi(x)$.

Application of CE for the solution for free particle with momentum $\vec{p}$ leads to:

$$j^\mu = |N|^2 p^\mu \qquad (2.4)$$

For the solution with the negative energy ($E = -\sqrt{\vec{p}^2 + m^2} \left(= p^0\right) < 0$) it leads to $\rho < 0$!

Hence it means a negative probability of particle presence. Let us assume that particle has the charge **–*e*** and let us make the substitution:

$$j^\mu \quad \rightarrow \quad -e \cdot j^\mu$$

In the following under the 4-vector current density we will understand:

$$j^\mu = -ie\left(\varphi^*\partial^\mu\varphi - \varphi\partial^\mu\varphi^*\right) \qquad (2.5)$$

Interpretation of the solution with E< 0 *(Pauli a Weiskopf)*

For free particle with momentum $\vec{p}$, energy E *(>0)* and charge $-e$ we have:

$$j^{\mu} = -e|N|^2 p^{\mu}$$

for anti-particle with momentum $\vec{p}$, energy E *(>0)* and charge e we have:

$$j^{\mu} = e|N|^2 p^{\mu} = -e|N|^2(-E,-\vec{p}) \tag{2.6}$$

Hence the current density for anti-particle with a given $\vec{p}$ and E *(>0)* agree with the current density for particle with momentum $-\vec{p}$ and energy $-E$.

Interpretation: Emission of antiparticle with energy E by some system is equivalent to absorption of particle with energy $-E$ by this system.

$$\begin{matrix} \uparrow\ e^{+} \\ |\ E>0 \end{matrix} \quad \equiv \quad \begin{matrix} |\ e^{-} \\ \downarrow E<0 \end{matrix} \qquad \uparrow \check{c}as \tag{2.7}$$

Else the solutions for particle with $E<0$ moving in time backward, describe anti-particle with $E>0$ moving in time forward.

Reason: the factor e^{-iEt} describing evolution of system stationary one can write:

$$e^{-iEt} = e^{-i(-E)(-t)} \tag{2.8}$$

4.3 Non/relativistic perturbation theory

Let us consider, in frame of the QM approach, a free particle enclosed in a volume. The quantum states of the particle are found as a solution of Schrödinger equation (SchR):

$$\begin{aligned} &H_0\varphi_n = E_n\varphi_n \qquad \text{pritom} \quad \int \varphi_m^*(x)\cdot\varphi_n dx = \delta_{mn} \\ &\varphi(x) = \varphi(\vec{x},t) = \varphi_n(\vec{x})\cdot e^{-iE_n t} \end{aligned} \tag{3.1}$$

If particle is moving in some force field ($V(\vec{x},t)$), then it is needed to look for solution of SchR:

$$i\frac{\partial\psi(x)}{\partial t} = (H_0 + V(x))\psi(x) \tag{3.2}$$

As the $\{\varphi_n\}$ is complete system of functions solution of Eq. (3.2) can be looked for in the form:

$$\psi(\vec{x},t) = \sum_n a_n(t) \cdot \varphi_n(\vec{x}) \cdot e^{-iE_n t} \qquad (3.3)$$

Applying $\int d^3\vec{x} \cdot \varphi_f^*(\vec{x}) \cdot$ to Eq. (3.2) and expressing ψ by means of (3.3) one gets:

$$\frac{da_f(t)}{dt} = -i\sum_n a_n(t) \cdot \int d^3\vec{x}\, \varphi_f^*(\vec{x}) V(\vec{x},t) \varphi_n(\vec{x}) \cdot e^{-i(E_n - E_f)t} \qquad (3.4)$$

where $a_n(t)$ is the probability amplitude of particle presence (in time *t*) in state $|n\rangle$,

$V_{fn} = \int d^3\vec{x}\, \varphi_f^*(\vec{x}) V(\vec{x},t) \varphi_n(\vec{x})$ is the particle transition probability from state $|n\rangle$ to state $|f\rangle$.

Let us suppose that the potential is small and is switched on at the time ***t=-T/2*** and is switch off at ***t=T/2*** and at the same time the particle was at the beginning in state $|i\rangle (\equiv \varphi_i)$:

$$a_k(-T/2) = \begin{cases} 1 & k = i \\ 0 & k \neq i \end{cases} \qquad (3.5)$$

If the potential is small and lasts only shortly, then conditions (3.5) are in 1st approximation, fulfilled during the whole interval (-T/2, T/2) and from (3.4) and (3.5) it follows:

$$\frac{da_f(t)}{dt} = -i\int d^3\vec{x}\, \varphi_f^*(\vec{x}) V(\vec{x},t) \varphi_i(\vec{x}) \cdot e^{-i(E_i - E_f)t} \qquad (3.6)$$

Having integrated (3.6) and using denotation $x \equiv (t, \vec{x})$ one gets:

$$T_{fi} = a_f(T/2) = -i\int d^4x\, \varphi_f^*(x) V(x)\, \varphi_i(x) \qquad (3.7)$$

Let us suppose that the potential does not depend on time $V(x) = V(\vec{x})$ then

$$\begin{aligned} T_{fi} &= -i\underbrace{\int d^3\vec{x}\, \varphi_f^*(\vec{x}) V(x)\, \varphi_i(\vec{x})}_{V_{fi}} \cdot \underbrace{\int_{-\infty}^{\infty} dt\, e^{-i(E_f - E_i)t}}_{2\pi\delta(E_f - E_i)} \cdot \\ &= 2\pi \cdot V_{fi} \cdot \delta(E_f - E_i) \end{aligned} \qquad (3.8)$$

Fermi rule. The transition probability from state $|i\rangle$ to state $|f\rangle$ is given by:

$$W_{fi} = \lim_{T\to\infty} \frac{|T_{fi}|^2}{T} = 2\pi |V_{fi}|^2\, \delta(E_f - E_i) \qquad (3.9)$$

Higher orders of perturbative theory

If in the relation (3.4) for $da_f(t)/dt$ we express $a_n(t)$ by means of (3.7), hence we use for it the 1st approximation, then we get:

$$\frac{da_f(t)}{dt} = \underbrace{(\cdots)_1}_{1.\,približenie} + (-i)^2 \sum_{n \neq i} V_{ni} \int_{-\infty}^{t} dt' e^{-i(E_i - E_n)t'} V_{fn} e^{-i(En - E_f)t} \tag{3.10}$$

The transition amplitude the reads

$$T_{fi} = (\cdots)_1 + (-i)^2 \sum_{n \neq i} \int_{-\infty}^{\infty} dt\, e^{-i(E_n = E_f)t} \cdot V_{fn} \cdot V_{ni} \cdot \int_{-\infty}^{t} dt' e^{-i(E_i - E_n)t} \tag{3.11}$$

To keep the integral through dt' limited (at integration to -∞) we added to integrand $e^{-t'}$:

$$\int_{-\infty}^{t} dt' \exp(-i(E_i - E_n)t') \rightarrow \int_{-\infty}^{t} dt' \exp(-i(E_i - E_n)t') = i \frac{\exp(-i(E_i - E_n + i\varepsilon)t)}{E_i - E_n + i\varepsilon} \tag{3.12}$$

For the transition amplitude it gives:

$$T_{fi} = -2\pi i \left(V_{fi} + \sum_{n \neq i} V_{fn} \cdot \frac{1}{E_n - E + i\varepsilon} \cdot V_{ni} \right) \delta(E_f - E_i) \tag{3.13}$$

The transition from the state $|i\rangle$ to state $|f\rangle$ can be depicted as is shown in Fig. 1. The second term of the perturbative expansion can be interpreted as follows: At the beginning particle is in the stave $|i\rangle$ and, due to presence of field, in a certain point and time it will interact and transit to an intermediate state $|n\rangle$, in which will be propagating until the moment when it reintegrate once-again and transits to the final state $|f\rangle$.

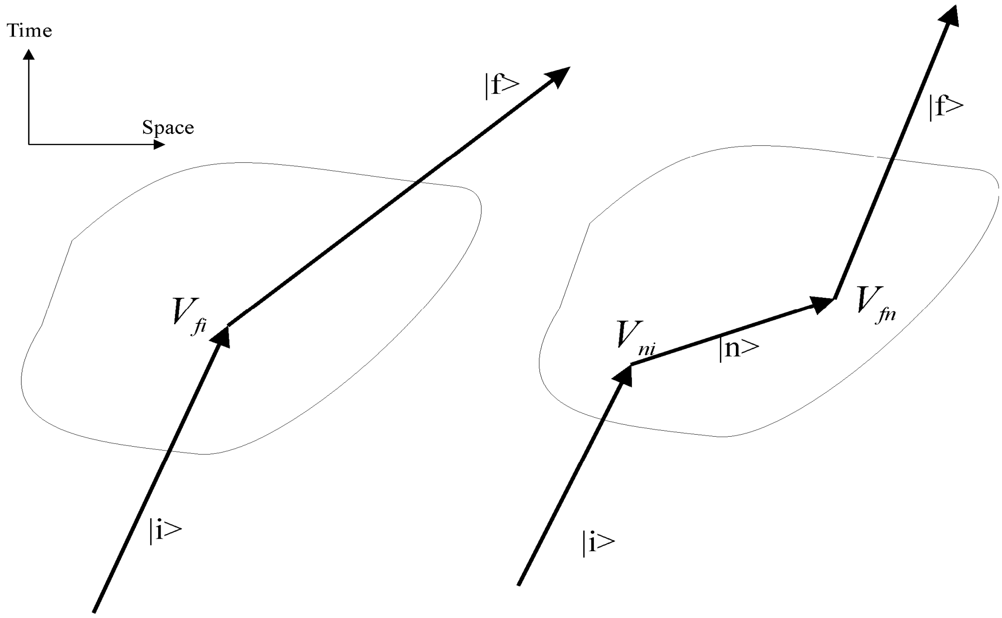


Fig. 1: Scattering of a particle on the static potential - 1st and 2nd order of perturbative approach.

The interpretation of the first term of the perturbative expansion is direct. The higher orders of the perturbative expansion can be obtained in such a way that into the relation (4) we replace $\boldsymbol{a_n(t)}$ by its perturbative expansion.

In general the above mentioned scattering can be characterized by the following:

- Interaction vertex is characterized by the factor $\boldsymbol{V_{ni}}$.
- Intermediate state $|\boldsymbol{n}\rangle$ describes propagator $-\frac{1}{\boldsymbol{E_i - E_n + i\varepsilon}}$; for the virtual state the energy conservation law is not valid ($\boldsymbol{E_n \neq E_i}$).
- The law of energy conservation is valid for the initial and final state $\rightarrow \boldsymbol{\delta(E_f - E_i)}$.

This perturbative theory does not contain:

- Diagrams in Fig. 1 present scattering on static potential and we are interested in scattering of particle in the field of other particle.
- It is needed to describe interaction of anti-particle.

4.4 Particle with spin S=0 in electromagnetic field

The basic assumptions are the following:

- Electromagnetic field is described by 4-potencial: $A^{\mu} = \left(A^0, \vec{A}\right)$.
- Particle has spin ***S=0***, momentum $\vec{p}$ and charge ***–e.***

From theory of electromagnetic field we have:

Motion of charge particle in the field A^{μ} we can get from free motion by the replacement:

$$p^{\mu} \to p^{\mu} - eA^{\mu}, \text{ resp. v QM: } \quad i\partial \to i\partial + eA^{\mu}.$$

The equation of motion of charged particle in the field A^{μ} is:

$$\left(\partial_{\mu}\partial^{\mu} + m^2\right)\varphi = \underbrace{\left(ie\left(\partial_{\mu}A^{\mu} + A_{\mu}\partial^{\mu}\right) + e^2 A^2\right)}_{-V}\varphi \qquad (4.1)$$

First order of prerturbative theory:

- The studied particle is at the beginning in state φ_i and due to its interaction with the field passes to the state φ_f;
- In potential ***V*** the terms ***~e²*** are neglected.

Under these conditions the transition element reads:

$$\begin{aligned} T_{fi} &= -i\int dx\, \varphi_f^*(x) V(x) \varphi_i(x) = i\int dx\, \varphi_f^*(x)(-ie)\left(\partial_{\mu}A^{\mu} + A^{\mu}\partial_{\mu}\right)\varphi_i(x) \\ &= -i\int (-ie)\left[\varphi_f^* \partial_{\mu}\left(A^{\mu}\varphi_i\right) + \varphi_f^*\left(\partial_{\mu}\varphi_i\right)A^{\mu}\right]dx \\ &= -i\int dx\,(-ie)\underbrace{\left[-\left(\partial_{\mu}\varphi_f^*\right)\varphi_i + \varphi_f^*\left(\partial_{\mu}\varphi_i\right)\right]}_{j_{\mu}(x)} A^{\mu} \\ &= -i\int dx\, j_{\mu}(x) A^{\mu}(x) \end{aligned} \qquad (4.2)$$

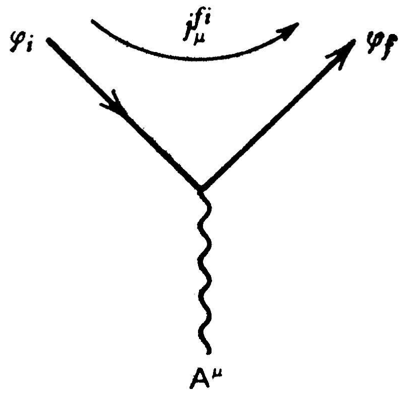
At the derivation we have used:

$$\partial_{\mu}\left(\varphi_f^* A^{\mu}\varphi_i\right) = \left(\partial_{\mu}\varphi_f^*\right)A^{\mu}\varphi_i + \varphi_f^*\partial_{\mu}\left(A^{\mu}\varphi_i\right) \qquad \text{and the fact that } \int dx\, \partial_{\mu}(\cdots) = 0$$

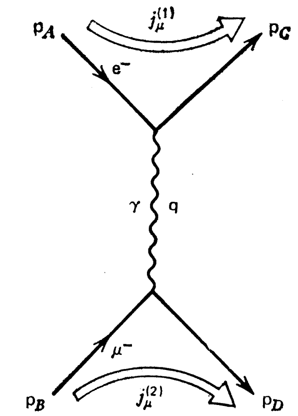
The current connected with transition of particle from state φ_i to state φ_f:

$$\begin{aligned} j_{\mu}^{fi}(x) &= -ie\left[\varphi_f^*(x)\partial_{\mu}\varphi_i(x) - \left(\partial_{\mu}\varphi_f^*(x)\right)\varphi_i(x)\right] \\ &= = eN_i N_f \left(p_i + p_f\right)_{\mu} \cdot \exp\left(i\left(p_f - p_i\right)x\right) \end{aligned} \qquad (4.3)$$

Fig. 2: Graphical representation of amplitude T_{fi}:
Particle with spin 0 scatters on static potential A^{μ}.



Scattering of two particles with spin 0



We look at the scattering of two particles of spin 0 in such a way that the first of the moves in the field of the second one. At the same time the potential, $\boldsymbol{A}^{\boldsymbol{\mu}}_{(2)}$, generated by the second particle reads:

$$\nabla^2 \boldsymbol{A}^{\boldsymbol{\mu}}_{(2)}(\boldsymbol{x}) = \boldsymbol{j}^{\boldsymbol{\mu}}_{(2)}(\boldsymbol{x}) \tag{4.4}$$

where

$$\boldsymbol{j}^{\boldsymbol{\mu}}_{(2)}(\boldsymbol{x}) = -\boldsymbol{e}\boldsymbol{N}_2 \boldsymbol{N}_4 \left(\boldsymbol{p_B} + \boldsymbol{p_D}\right)^{\boldsymbol{\mu}} \boldsymbol{e}^{-i(\boldsymbol{p_D}-\boldsymbol{p_B})\boldsymbol{x}}$$

The solution for the potential is:

$$\boldsymbol{A}^{\boldsymbol{\mu}}_{(2)}(\boldsymbol{x}) = -\frac{1}{\boldsymbol{q}^2} \boldsymbol{j}^{\boldsymbol{\mu}}_{(2)}(\boldsymbol{x}) , \quad \boldsymbol{q} = \boldsymbol{p_D} - \boldsymbol{p_B} \tag{4.5}$$

Finally the amplitude of scattering of two particles of spin 0 (e.g. $\boldsymbol{\pi}$ and $\boldsymbol{K}$) reads:

$$T_{fi} = -i\int d^4x \, j^{(1)}_{\mu}(x) A^{\mu}_{(2)}(x) = -i\int d^4x \; j^{(1)}_{\mu}(x) \, \frac{1}{q^2} \, j^{\mu}_{(2)}(x) = -i\int d^4x \; j^{\mu}_{(1)}(x) \, \frac{g_{\mu\nu}}{q^2} \, j^{\upsilon}_{(2)}(x) \tag{4.6}$$

Where $j^{(1)}_{\mu}(x) \left(j^{(2)}_{\mu}(x)\right)$ is current density connected with the transition $\boldsymbol{\pi}$($\boldsymbol{K}$) from initial to final state ($|i\rangle \rightarrow |f\rangle$) . The quantitie $\frac{g_{\mu\nu}}{q^2}$ is propagator photon.

If we express the $j_\mu^{(1)}(x)$ $\left(j_\mu^{(2)}(x)\right)$ in the relation (4.6) by means (4.4) and carried out integration – we get:

$$T_{fi} = -iN_1 N_2 N_3 N_4 \cdot (2\pi)^4 \delta^4 (p_1 + p_2 - p_3 - p_4) \cdot M_{fi} \quad (4.7)$$

where

$$-iM_{fi} = ie(p_1 + p_3)^\mu \left(-i\frac{g_{\mu\nu}}{q^2}\right) ie(p_2 + p_4)^\nu \quad (4.8)$$

Remark: Note the symmetry between both particles (***π*** and ***K***).

Normalization: One of the possible normalizations is the normalization ***per 1 particle in the volume V***.

$$\varphi(t,\vec{x}) = N \cdot e^{-ipx} = \frac{1}{\sqrt{2E \cdot V}} e^{-ipx} \quad (4.9)$$

Usually it is taken ***V=1***. Applying the continuation equation we get:

$$\int_V d^3\vec{x}\,\rho = 1$$

4.5 The relation between cross section and transition amplitude

The relation between experimentally measured quantity (cross section) and quantity given by theory (transition amplitude):

$$d\sigma = \frac{W_{fi}}{|\vec{v}|\rho_1\rho_2} \cdot d\Omega \quad (5.1)$$

where

- W_{fi} is the transition probability per unit time:

$$W_{fi} = \frac{|T_{fi}|^2}{VT} = \frac{|M_{fi}|^2}{2E_1 2E_2 2E_3 2E_4 \cdot V^4} (2\pi)^4 \delta^4 (p_1 + p_2 - p_3 - p_4) \quad (5.2)$$

- The density of initial states:

$$|\vec{v}|\rho_1\rho_2 = \frac{|\vec{v}|}{V^2} \quad (5.3)$$

- $d\Omega$ je počet konečných stavov:

$$d\Omega = \frac{Vd^3 p_3}{(2\pi)^3} \frac{Vd^3 p_4}{(2\pi)^3} \quad (5.4)$$

After putting the (5.3-5) into (5.2) we get:

$$d\sigma = \left|M_{fi}\right|^2 \frac{1}{|\vec{v}| 2E_1 2E_2} \cdot \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_3} \cdot (2\pi)^4 \delta^4 (p_1 + p_2 - p_3 - p_4) \qquad (5.5)$$

Analogically for the decay rate ($A \to 1+2+\ldots+n$):

$$d\Gamma = \left|M_{A\to 1\cdots n}\right|^2 \frac{1}{2E_A} \cdot \frac{d^3 p_1}{(2\pi)^3 2E_1} \cdots \frac{d^3 p_n}{(2\pi)^3 2E_n} \cdot (2\pi)^4 \delta^4 (p_A - p_1 \cdots - p_n) \qquad (5.6)$$

Two particle phase space

In case of the 2-particle final state phase space (after inclusion δ-function) we have:

$$dLips_2(P) = \frac{Vd^3 p_3}{2E_3 (2\pi)^3} \frac{Vd^3 p_4}{2E_4 (2\pi)^3} (2\pi)^4 \delta^4 (P - p_3 - p_4) =$$

$$= \frac{1}{(2\pi)^2} d^4 p_3\, \delta\left(p_3^2 - m_3^2\right) \Theta(p_3^0) \cdot d^4 p_4\, \delta\left(p_4^2 - m_4^2\right) \; \Theta(p_4^0) \cdot \delta^4 (P - p_3 - p_4) \qquad (5.7)$$

where m_3 and m_4 are the masses of output particles and $P = p_1 + p_2$ is the full momentum of input particles.

We used the properties of δ-function:

$$\int g(x) \delta(f(x)) dx = \int g(x) \frac{1}{|f'(x)|} \delta\left(x - \sum_i x_i \right) \qquad (5.8)$$

Where $\boldsymbol{x_i}$ are the zero points of the function $\boldsymbol{f(x)}$

After integrating (5.7) through $d^4 p_4$ and the subsequent integration through dp_3^0 we get:

$$dLips_2(P) = \frac{1}{(2\pi)^2} d^4 p_3\, \delta\left(p_3^2 - m_3^2\right) \cdot \delta\left((P - p_3)^2 - m_4^2\right) \Theta(p_3^0)$$

$$\xrightarrow[\text{int } p_3^0]{} \frac{1}{(2\pi)^2} \frac{d^3 p_3}{2E_3} \cdot \delta\left((P - p_3)^2 - m_4^2\right) \qquad (5.9)$$

After going to spherical coordinates ($d^3 p_3 = p^2 dp\, d\cos\theta\, d\varphi$,

$p = \sqrt{\left(p_3^1\right)^2 + \left(p_3^2\right)^2 + \left(p_3^3\right)^2}$) finally we get:

$$dLips_2(P) = \frac{1}{(2\pi)^2} d\cos\theta\, d\varphi\, dp \frac{p^2}{2\sqrt{p^2 + m_3^2}} \cdot \delta\left(M^2 - 2M\sqrt{p^2 - m_3^2} + m_3^2 - m_4^2\right) =$$

$$= \frac{1}{(2\pi)^2} d\cos\theta\, d\varphi\, \frac{\lambda^{1/2}\left(M^2, m_3^2, m_4^2\right)}{8M^2} \qquad (5.9)$$

where $\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$