Spinor field - Dirac equation

ver *23. 11. 2007*

The Dirac equation (DR) describes classical spinor field and the quanta of this field are particles of spin ½ - similarly as photons represent quanta of electromagnetic field. Such particles (s = ½) are characterized by:

$$\underbrace{x \equiv (t, \vec{x})}_{\text{4-vector of position}}\ ,\quad \underbrace{p \equiv (E.\vec{p})}_{4-momentum},\quad \underbrace{s_z\,(=-\tfrac{1}{2},\tfrac{1}{2})}_{\text{spin projection}},\quad \underbrace{e}_{\text{charge}}$$

The form of DR:

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\Psi(x) = 0 \qquad (1a)$$

$$i\partial_{\mu}\bar{\Psi}(x)\ \ \gamma^{\mu} + m\bar{\Psi}(x) = 0 \qquad (1b)$$

where

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}\ \partial_{\mu} \equiv (\partial_t, \nabla) \qquad (2)$$

$\Psi(x)$ is the 4-component spinor and $\bar{\Psi} = \Psi^{+}\gamma^{0}$ is the Dirac-conjugated spinor and $\vec{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices (see Appendix A).

Current density and equation of continuity

The current density for particle of spin ½ and charge $-e$ reads:

$$j^{\mu}(x) = -e\bar{\Psi}(x)\,\gamma^{\mu}\,\Psi(x) \qquad (3)$$

and fulfills the continuity equation:

$$\partial_{\mu} j^{\mu}(x) = 0 \qquad (4)$$

The solution of DR for free particle with momentum $\vec{p}$ and spin 1/2 can be expressed in the form

$$\Psi_{\vec{p}}(x) = u(\vec{p})\exp(-ipx) \qquad (5)$$

Where 4-component spinor $u(\vec{p})$ fulfills equation

$$\left(\gamma^{\mu} p_{\mu} - m\right) u(\vec{p}) = 0 \qquad (6)$$

For free particle with momentum $\vec{p}$ Eq. (6) has *4* independent solutions:

Two solutions with $E > 0$ and two solutions with $E < 0$.

The solutions with positive energy ($p_0 = E > 0$):

$$u^{(i)}(\vec{p}) = N\begin{pmatrix} \varphi^{(i)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\varphi^{(i)} \end{pmatrix} \quad i = 1,2 \qquad \varphi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \varphi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7a}$$

The solutions with negative energy ($p_0 = -E < 0$) are:

$$u^{(i)}(\vec{p}) = N\begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(i)} \\ \chi^{(i)} \end{pmatrix} \quad i = 3,4 \quad \chi^{(3)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi^{(4)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7b}$$

Where N is the normalization constant (characterizes particle density – see (3)).

Interpretation of the solutions: in state with momentum $\vec{p}$ particle acquires energy $\pm E \left(= \sqrt{\vec{p}^2 + m^2}\right)$ ($-E$ corresponds to anti-particle), and in both cases spin projection can acquire two value with opposite sin (oriented in direction of motion and against it).

On the solution of Dirac equation

First 2 solutions $u^{(1,2)}(\vec{p})e^{-ipx}$ describes electron with energy E and momentum $\vec{p}$.

The next 2 solutions $u^{(3,4)}(\vec{p})e^{-ipx}$ negative energy corresponds to positron. However a positron with energy E and momentum $\vec{p}$ will be described by the solution for electron with $-E$ and $-\vec{p}$, therefore one can write:

$$u^{(3,4)}(-\vec{p})e^{-i(-p)x} \equiv v^{(2,1)}(\vec{p})e^{ipx} \tag{7.1}$$

The change of index order: $(3,4) \to (2,1)$ follows from the fact that the simultaneous change of direction of spin and momentum will not change helicity ($(1/2)\ \vec{\sigma}\cdot\vec{p}$). As the change of index order changes direction of spin, the positron would have not only opposite momentum but also spin (if compared with electron) , however if we make a replacement $3\to2$ and $4\to1$, both the positron and electron will have defined helicity in the same way.

The DR for spinors $u(\vec{p})$ and $v(\vec{p})$ evidently reads:

$$(\hat{p}-m)u(\vec{p}) = 0\ , \qquad (\hat{p}+m)v(\vec{p}) = 0 \quad \text{kde} \quad \hat{p} \equiv \gamma^{\mu}p_{\mu} \tag{7.2}$$

Normalization of spinor function is important for the correct determination of relation between cross section and process amplitude – usually it is done for:

- 1 one particle in unit volume,
- 2E in unit volume.

Such a choice leads to the following values of the normalization constant N:

$$\int_{unitVol} \rho dV = \int \psi^{+}\psi \ dV = u^{+}u = \begin{cases} 2E \\ 1 \end{cases} \rightarrow \begin{matrix} N = \sqrt{E+m} \\ N = \sqrt{E+m} \Big/ \sqrt{2E} \end{matrix} \tag{7.3}$$

Relation of completeness. These relations are very important at calculation of amplitude:

$$\begin{aligned} \sum_{s=1,2} u^{(s)}(p)\bar{u}^{(s)}(p) &= \hat{p} + m \\ \sum_{s=1,2} v^{(s)}(p)\bar{v}^{(s)}(p) &= \hat{p} - m \end{aligned} \tag{7.4}$$

Weyl's representation of γ-matrices and solution of DR

The structure of γ-matrices in this representation is the following:

$$\gamma^{0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^{i} = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \Rightarrow \quad \gamma^{5} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.5}$$

The field function we will write as follows:

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \ . \tag{7.6}$$

Where ψ_R a ψ_L are 2-component spinors.

The DR has got in this case the form:

$$\left(\gamma^{\mu} p_{\mu} - m\right)\psi = 0 \quad \Rightarrow \quad \begin{pmatrix} -m & p_0 + \vec{\sigma}\cdot\vec{p} \\ p_0 - \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = 0\,, \tag{7.7}$$

and gives the following solutions:

$$\psi_R = \frac{p_0 + \vec{\sigma}\cdot\vec{p}}{m}\psi_L$$

$$\psi_L = \frac{p_0 - \vec{\sigma}\cdot\vec{p}}{m}\psi_R$$

An interesting case occurs at $m = 0$. In this case ψ_R a ψ_L are eigenstates of $\vec{\sigma}\cdot\vec{p}$-operator (is proportional projection of spin to direction of motion):

$$\vec{\sigma}\cdot\vec{p}\,\psi_R = p_0\,\psi_R \quad \text{a} \quad \vec{\sigma}\cdot\vec{p}\,\psi_L = -p_0\,\psi_L$$

In relativistic case one can write***:***

$$\psi_R \text{ is big } (\psi_R \gg \psi_L) \text{ for } \vec{\sigma}\cdot\vec{p} > 0 \text{ and } p_0 > 0$$

$$\psi_L \text{ is big } (\psi_L \gg \psi_R) \text{ for } \vec{\sigma}\cdot\vec{p} < 0 \text{ and } p_0 > 0$$

In ultra-relativistic case $\vec{\sigma}\cdot\vec{p}/p_0 = \vec{\sigma}\cdot\hat{\vec{p}}$ is operator of helicity and indexes R and L correspond to the right and left solutions of DR, respectively.

Interaction of particle of spin ½ with electromagnetic field

The equation of motion for particle of spin ½ in electromagnetic field we get from Dirac equation by the replacement: $p^\mu \to p^\mu - QeA^\mu$, where Q is charge of particle expressed in elementary charges (e) (for electron $Q = -1$). For electron one gets:

$$(\gamma_\mu p^\mu - m)\psi = \gamma^0 V\psi\,, \qquad \gamma^0 V = -e\gamma_\mu A^\mu \qquad (8)$$

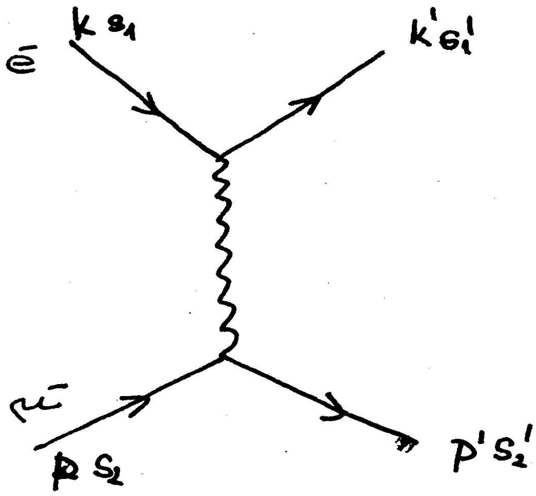
γ^0 is taken out of V to get at transition to the non-relativistic case the Schrodinger equation. The equation (8) we solve in an analogical way as in the case of particle with spin 0 by using the perturbative method.

The solution in the first order of perturbative method can be demonstrated by the case of $e\mu$-scattering.

Rozptyl $e^-\mu^- \to e^-\mu^-$

Let us treat scattering of an e^- with 4-momentum k on muon with 4-momentum p (*see Fig. 1).* The perturbative method gives for the element of transition from initial to final state: $T_{fi} = -i\int \psi_f^+(x)V(x)\psi_i\,dx = -i\int(-e)\overline{\psi}_f(x)\gamma_\mu\psi_i A^\mu(x)dx$.

If we look at this problem in such a way that the electron undergoes a scattering in the potential created by muon for the ***eμ***-scattering we get:



Initial state: $|i\rangle = |ks_1, ps_2\rangle$

Final state: $|f\rangle = |k's_1', p's_2'\rangle$

Amplitude of transition:

$$T_{fi} = -i\int j_\mu^{(e)}(x)\left(-\frac{1}{q^2}\right) j_{(m)}^\mu(x)\,dx = \\ = -i\left(2\pi\right)^4 \delta^{(4)}(k + p - k' - p') \quad M_{fi} \tag{9}$$

where

$$j_{(l)}^\mu = -\overline{\psi}_f \gamma^\mu \psi_i = -e\bar{u}_f \gamma^\mu u_i\, e^{i(p_f - p_i)x}$$

***Fig. 1*: Scattering of electron on muon.**

The matrix element M_{fi} (after putting the expression for current into Eq. (9)) reads:

$$M_{fi} = -e^2\bar{u}(k', s_1') \quad \gamma^\mu u(k, s_1) \cdot \frac{1}{q^2} \cdot \bar{u}(p', s_2') \quad \gamma_\mu u(p, s_2) \tag{10}$$

And we will interested in the non-polarized cross section, i.e. we will suppose that in the initial state both electron and muon will acquire with the same probability the two values of spin projection and $\left|M_{fi}\right|^2$ will be averaged through the initial states and summed through the final spin states.

$$\overline{\left|M_{fi}\right|^2} = \frac{e^4}{q^4} \cdot L_{(el)}^{\mu\nu}\, L_{\mu\nu}^{(mu)} \tag{11}$$

Where

$$L_{(el)}^{\mu\nu} = \frac{1}{2}\sum_{s_1 s_1'} \bar{u}(k', s_1')\gamma^\mu u(k, s_1) \cdot \left(\bar{u}(k', s_1')\gamma^\nu u(k, s_1)\right)^* \\ = \frac{1}{2} Tr\left[\left(\hat{k}' + m\right) \ \gamma^\mu \left(\hat{k} + m\right) \ \gamma^\nu\right], \ \hat{k} = k_\alpha \gamma^\alpha \tag{12}$$

It is easily to show $\left(u_i \equiv u(k, s_1) \text{ a } u_f \equiv u(k', s_1')\right)$ that

$$\left(\bar{u}_f \gamma^\nu u_i\right)^* = \left(u_f^+ \gamma^0 \gamma^\nu u_i\right)^+ = u_i^+ \gamma^{\nu+} \gamma^0 u_f = u_i^+ \gamma^0 \gamma^\nu u_f = \bar{u}_i \gamma^\nu u_f$$

or $\gamma^{\nu+}\gamma^0 = \gamma^0 \gamma^\nu$ (see properties of γ-matrices).

$$L^{\mu\nu}_{(el)}=\frac{1}{2}\sum_{s_1 s_1'}\bar{u}_f\gamma^{\mu}u_i\cdot\bar{u}_i\gamma^{\nu}u_f=\frac{1}{2}\sum_{s_1'}\bar{u}_f\gamma^{\mu}\left(\hat{k}+m\right)\gamma^{\nu}u_f=$$
$$=\frac{1}{2}\sum_{s_1'}\left(\bar{u}_f\right)_i\left(\gamma^{\mu}\left(\hat{k}+m\right)\gamma^{\nu}\right)_{ij}\left(u_f\right)_j=\frac{1}{2}\sum_{s_1'}\left(u_f\right)_j\left(\bar{u}_f\right)_i\left(\gamma^{\mu}\left(\hat{k}+m\right)\gamma^{\nu}\right)_{ij}=$$
$$=\frac{1}{2}\left(\hat{k}'+m\right)_{ji}\left(\gamma^{\mu}\left(\hat{k}+m\right)\gamma^{\nu}\right)_{ij}=\frac{1}{2}Tr\left[\left(\hat{k}'+m\right)\gamma^{\mu}\left(\hat{k}+m\right)\gamma^{\nu}\right]\tag{12}$$

Where we used the completeness relations: $\sum_{s_1=1,2}u_i\bar{u}_i=\hat{k}+m$.

A similar expression is valid for $L^{(mu)}_{\mu\nu}$. For calculation of process matrix element is needed to figure out the trace of a product of γ-matrices.

$$L^{\mu\nu}_{(el)}=\frac{1}{2}Tr\left[\hat{k}'\gamma^{\mu}\hat{k}\gamma^{\nu}+m^2\gamma^{\mu}\gamma^{\nu}\right]=\frac{k'_{\alpha}k_{\beta}}{2}Tr\left[\gamma^{\alpha}\gamma^{\mu}\gamma^{\beta}\gamma^{\nu}\right]+\frac{m^2}{2}Tr\left[\gamma^{\mu}\gamma^{\nu}\right]\tag{13}$$

Using the relations for traces of γ-matrices (see appendix A):

$$Tr\left[\gamma^{\alpha}\gamma^{\mu}\gamma^{\beta}\gamma^{\nu}\right]=2\left(g^{\alpha\mu}g^{\beta\nu}-g^{\alpha\beta}g^{\mu\nu}+g^{\alpha\nu}g^{\mu\beta}\right),\qquad Tr\left[\gamma^{\mu}\gamma^{\nu}\right]=2g^{\mu\nu}$$

We get:

$$L^{\mu\nu}_{(el)}=2\left(k'^{\mu}k^{\nu}-\left(k'\cdot k\right)g^{\mu\nu}+k'^{\nu}k^{\mu}+m^2g^{\mu\nu}\right)\tag{14}$$

And analogically for muon tensor we get:

$$L^{\mu\nu}_{(muon)}=2\left(p'^{\mu}p^{\nu}-\left(p'\cdot p\right)g^{\mu\nu}+p'^{\nu}p^{\mu}+M^2g^{\mu\nu}\right)\tag{15}$$

Using Eq. (11) it leads us to the result

$$\overline{|M|^2}=\frac{8e^4}{q^4}\cdot\left[(k'\cdot p')(k\cdot p)+(k'\cdot p)(k\cdot p')-m^2(p'\cdot p)-M^2(k'\cdot k)+2m^2M^2\right]\tag{16}$$

where m (M) is mass of electron (muon).

Recipe for construction of matrix element

Amplitude (matrix element) of eμ-scattering we can express as follows:

$$M_{fi}=-i\bar{u}(k',s_1')\left(-ie\gamma^{\mu}\right)u(k,s_1)\cdot\frac{-ig_{\mu\nu}}{q^2}\cdot\bar{u}(p',s_2')\ \left(-ie\gamma^{\nu}\right)u(p,s_2)\tag{17}$$

If we assign the factors to the individual parts of the diagram of ***eμ***-scattering as is shown in Fig. 2, the process matrix element can be easily found.

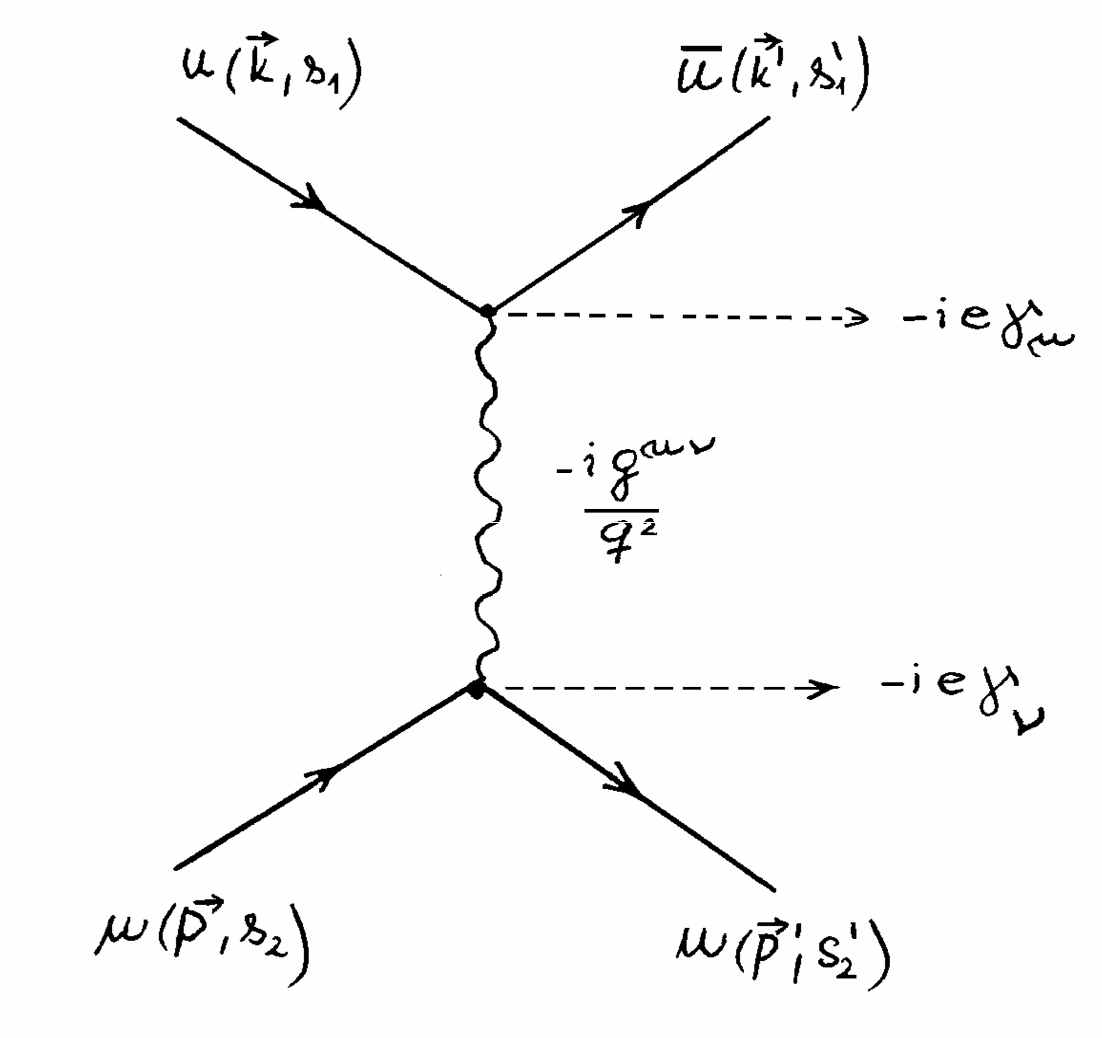


Fig. 2: Electron - muon scatering - e and μ are exchanging a virtual photon.

Hence for construction of a process matrix element we need to assign to:

Interaction vertex $\Rightarrow -ie\gamma_\mu$

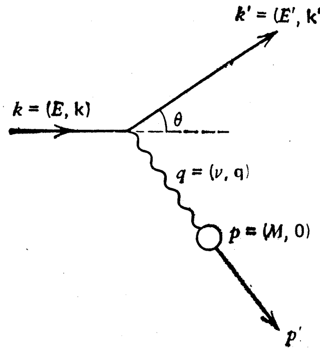
Input particle with spin ½ $\Rightarrow u(\vec{p}, s)$

Output particle with spin ½ $\Rightarrow \bar{u}(\vec{p}, s)$

Propagator of photon $\Rightarrow \dfrac{-ig^{\mu\nu}}{q^2}$

Scattering of $e^-\mu^- \rightarrow e^-\mu^-$ in laboratory system

Let us consider the $e^-\mu^-$ scattering in laboratory system (LS) – as is shown in Fig. *3*.



In LS muon is in a rest: $\boldsymbol{p} \equiv (\boldsymbol{M}, \vec{0})$, we know the state of motion of the incident electron $(\boldsymbol{E}, \vec{\boldsymbol{k}})$ and experimentally we measure the energy of output electron (***E'***) and angle of his declination from the original direction (***θ***).

Fig. 3: The eμ- scattering in lab. system

If we come from the general formula for ***eμ***-scattering (16), we neglect the terms proportional $\boldsymbol{m}^2$ ($m \equiv$ mass of electron) and use the approximation:

$$q^2 \approx -2k \cdot k' = -2EE'(1-\cos\theta) = -4EE' \cdot \sin^2\frac{\theta}{2} \tag{18}$$

And for the square of matrix element $\boldsymbol{M_{fi}}$ module one gets:

$$\overline{|M_{fi}|^2} = \frac{8e}{q^4} 2M^2 EE' \left\{ \cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2} \right\} \tag{19}$$

The cross section reads:

$$\frac{d\sigma}{dE' d\Omega} = \frac{(2\alpha E')^2}{q^4} \left\{ \cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2} \right\} \cdot \delta\left(\nu + \frac{q^2}{2M} \right) \tag{20}$$

where $\alpha = e^2/4\pi$, $\nu = E - E'$.

Or if we are interested only in the angle of scattered electron:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}} \frac{E'}{E} \cdot \left\{ \cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2} \right\} \tag{21}$$

The formulae for the scattering cross section of point-like particles (20) and (21) are very important, because a deviation from the law of scattering of point-like particles indicates the presence of a non-discrete structure, hence it provide us with an information about structure of particles.

Remark. If instead of muon we would take point-like particle with spin 0, the cross section for scattering to angle θ would read:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}} \frac{E'}{E} \cdot \cos^2\frac{\theta}{2} \tag{22}$$

From comparison of the relations (21) and (22) it is obvious that the term containing $\sin^2 \theta/2$ v (21) arises as a consequence arises of electron scattering on spin (magnetic moment) of muon.

Comparison of scattering of particle with spin 0 and ½

The scattering matrix element is, for both the cases, represented by formally the same relation:

$$T_{fi} = -i\int dx\, j_\mu^{fi}(x)\cdot A^\mu(x)$$

The difference is in the structures of their electromagnetic currents. Particles with spin 0 interacts with electromagnetic field exclusively through the charge e and structure of current (transition of particle from the state φ_i to state φ_f) reads

$$j_\mu^{fi}(x) = eN_i N_f \left(p_i + p_f\right)_\mu \cdot e^{=iqx} \qquad (23)$$

In case of particle spin ½ the structure of current is the following:

$$j_\mu^{fi}(x) = -e\bar{u}_f \gamma^\mu u_i e^{-iqx} \qquad (24)$$

Using Gordon expansion we get:

$$-e\bar{u}_f \gamma^\mu u_i = -e\bar{u}_f\left(\frac{\left(p_f + p_i\right)^\mu}{2m} - i\frac{\sigma^{\mu\nu}q_\nu}{2m}\right)u_i\,, \qquad \sigma^{\mu\nu} = \tfrac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \qquad (25)$$

We see that in the case of particle with spin ½ , in addition to interaction through charge $\sim(p_f+p_i)$ there is also an interaction corresponding to the term $\sigma^{\mu\nu}q_\nu$. This term describes the interaction through magnetic moment of electronu:

$$\vec{\mu} = -\frac{e}{2m}\vec{\sigma} = -g\frac{e}{2m}\vec{S} \qquad (26)$$

Where $\vec{S} = \vec{\sigma}/2$ a $g=2$ is gyromagnetic factor.

Hence electron (particle with spin 1/2) interacts with electromagnetic field not only through its charge but also through magnetic moment!

Remark. *For understanding of the fact that the second term in (25) represents interaction of magnetic moment, it is taken into account the following:*

- $q_0 = 0$ *due to the energy conservation law (* $E_i = E_f$ *)*

- *Space part of* $\sigma^{\mu\nu}$ *is* $\sigma^{ij} = \varepsilon^{ijk}\begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix} \quad i,j = 1,2,3$
- *Take only the upper components of the field functions:* $\psi^i(x)\left(= u^i(p)\exp(-ip_i x)\right)$ a $\psi^f(x)$.

Appendix A. Algebra of γ-matrices – their basic properties

Fundamental anti-commutator:

$$\left\{\gamma^\mu,\gamma^\nu\right\} = 2g^{\mu\nu}, \quad g^{\mu\nu} \equiv \textbf{ metric tensor} \tag{A.1}$$

γ-matrices in the standard representation:

$$\gamma^0 = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix}0 & \vec{\sigma} \\ -\vec{\sigma} & 0\end{pmatrix} \qquad \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} \tag{A.2}$$

where

$$\vec{\sigma} = (\sigma_1,\sigma_2,\sigma_3) \quad , \quad \sigma_1 = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} \quad \sigma_2 = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix} \quad \sigma_3 = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} \tag{A.3}$$

From the definition γ_5 or from its explicit expression it follows:

$$\left\{\gamma_5,\gamma^\mu\right\} = 0, \qquad \gamma_5^2 = 1 \tag{A.4}$$

And also is valid:

$$\gamma_5 \equiv \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \qquad \varepsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \textbf{even combinations } 0,1,2,3 \\ -1 & \textbf{odd combinations } 0,1,2,3 \\ 0 & \textbf{2 and identical indexes} \end{cases} \tag{A.5}$$

At calculation of Feynmann diagrams it is often needed to use the following ***properties of traces of γ-matrices products***:

$$\begin{aligned} Tr\left(\gamma^\mu\gamma^\nu\right) &= 4g^{\mu\nu} \\ Tr\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= 4\left[g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}\right] \\ Tr\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= 4i\varepsilon^{\mu\nu\rho\sigma} \\ Tr\left(\gamma_5\gamma^\mu\gamma^\nu\right) &= 0 \end{aligned} \tag{A.6}$$

$$Tr\left(\underbrace{\gamma^\mu\cdots\gamma^\sigma}_{neparné}\right) = 0$$

And is also valid ($\hat{a} \equiv a_\mu\gamma^\mu$):

$$\begin{aligned} \gamma_\mu \gamma^\mu &= 4 \\ \gamma_\mu \hat{a} \gamma^\mu &= -2\hat{a} \\ \gamma_\mu \hat{a}\hat{b} \gamma^\mu &= 4a \cdot b \\ \gamma_\mu \hat{a}\hat{b}\hat{c} \gamma^\mu &= -2\hat{c}\hat{b}\hat{a} \\ \hat{a}\hat{b} &= 2(ab) - \hat{b}\hat{a} \end{aligned} \tag{A.7}$$

Appendix B. Gordon expansion

Electromagnetic current evoked by transition of electron form the state $|i\rangle$ to $|f\rangle$

We can decompose to components:

$$-e\bar{u}_f \gamma^\mu u_i = -e\bar{u}_f \left(\frac{(p_f + p_i)^\mu}{2m} - i\frac{\sigma^{\mu\nu} q_\nu}{2m} \right) u_i \tag{B.1}$$

where $\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu\right)$.

Let us begin with

$$\begin{aligned} & i\bar{u}_f \left(\sigma^{\mu\nu} q_\nu\right) u_i = -\tfrac{1}{2}\bar{u}_f \left(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu\right) u_i = \\ & \underbrace{-\tfrac{1}{2}\bar{u}_f \gamma^\mu \gamma^\nu (p_f)_\nu u_i}_{V_1} \quad \underbrace{+\tfrac{1}{2}\bar{u}_f \gamma^\nu \gamma^\mu (p_f)_\nu u_i}_{V_2} \\ & \underbrace{+\tfrac{1}{2}\bar{u}_f \gamma^\mu \gamma^\nu (p_i)_\nu u_i}_{V_3} \quad \underbrace{-\tfrac{1}{2}\bar{u}_f \gamma^\nu \gamma^\mu (p_i)_\nu u_i}_{V_4} \end{aligned} \tag{B.2}$$

The terms V_2 and V_3 we can easily adjust using the Dirac equation:

$$V_2 = \tfrac{1}{2}\bar{u}_f \hat{p}_f \gamma^\mu u_i = \tfrac{1}{2} m \bar{u}_f \gamma^\mu u_i \qquad \left(\bar{u}_f \hat{p}_f = m\bar{u}_f\right) \tag{B.3}$$

$$V_3 = \tfrac{1}{2}\bar{u}_f \gamma^\mu \hat{p}_i u_i = \tfrac{1}{2} m \bar{u}_f \gamma^\mu u_i \qquad \left(\hat{p}_i u_i = m u_i\right) \tag{B.4}$$

For the terms V_1 and V_4 it is needed to use: $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$

$$\begin{aligned} V_1 &= -\tfrac{1}{2}\bar{u}_f \left(-\gamma^\nu \gamma^\mu + 2g^{\mu\nu}\right)(p_f)_\nu u_i = \tfrac{1}{2}\bar{u}_f \gamma^\nu (p_f)_\nu \gamma^\mu u_i - \bar{u}_f p_f^\mu u_i = \\ &= \tfrac{1}{2} m \bar{u}_f \gamma^\mu u_i - \bar{u}_f p_f^\mu u_i \end{aligned} \tag{B.5}$$

$$\begin{aligned} V_4 &= -\tfrac{1}{2}\bar{u}_f \left(-\gamma^\mu \gamma^\nu + 2g^{\mu\nu}\right)(p_i)_\nu u_i = \tfrac{1}{2}\bar{u}_f \gamma^\mu (p_i)_\nu \gamma^\nu u_i - \bar{u}_f p_i^\mu u_i = \\ &= \tfrac{1}{2} m \bar{u}_f \gamma^\mu u_i - \bar{u}_f p_i^\mu u_i \end{aligned} \tag{B.6}$$

Hence for B.2 using (A3-6) we get:

$$\bar{u}_f i\sigma^{\mu\nu}\left(p_f - p_i\right)_\nu = V_1 + V_2 + V_3 + V_4 = 2m\bar{u}_f \gamma^\mu u_i - \bar{u}_f \left(p_i + p_f\right)^\mu u_i \tag{B.7}$$

From this we get the B.1.

Electromagnetic field - photon

The equations of motion for the electromagnetic field are the Maxwell equations:

$$\partial_\mu F^{\mu\nu} = j^\nu \quad \text{a} \quad \partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{1}$$

Where

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho} F_{\mu\nu}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \tag{2}$$

$F^{\mu\nu}$ is the tensor of electromagnetic field, $A^\mu = (\varphi, \vec{A})$ is the 4-potencial and $\tilde{F}^{\mu\nu}$ is the dual tensor of electromagnetic field.

The explicit expression of $F^{\mu\nu}$ leads to the Maxwell equation for 4-potencial:

$$\partial^\nu \partial_\nu A^\mu - \partial^\mu (\partial_\nu A^\nu) = j^\mu \tag{3}$$

or

$$\left(g^{\mu\nu}\partial_\lambda \partial^\lambda - \partial^\mu \partial^\nu\right) A_\nu = j^\mu \tag{4}$$

The potential A^μ exhibits gauge degree of freedom that consist in the fact that physically are measurable the intensity of electric field $\vec{E}$ and induction magnetic field $\vec{B}$ that are defined as follows:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \varphi, \quad \vec{B} = \nabla \times \vec{A} \tag{5}$$

The quantities $\vec{E}, \vec{B}$ will not change if we make the gauge transformation 4-potencial A^μ:

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \chi \tag{6}$$

where χ is an arbitrary function differentiable in the 2nd derivatives. Hence the whole class of potentials leads to the same configuration of electromagnetic field. This enables us to choose A^μ in such a way that the Lorentz condition is fulfilled:

$$\partial^\nu \partial_\nu A^\mu = j^\mu \quad at \quad \underbrace{\partial_\mu A^\mu = 0}_{\text{Lorentz condition}} \tag{7}$$

Additional freedom. If potential A^μ fulfills the Lorentz condition it is still not determined by this condition unambiguously. If we go to an other potential

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda \tag{8}$$

where Λ fulfills condition $\partial_\mu \partial^\mu \Lambda = 0$. It is clear that if A_μ fulfills the Lorentz condition then it fulfills also A'_μ. Hence, also after fulfillment of Lorentz condition there is still a freedom in the choice of description of electromagnetic field.

Let us consider electromagnetic field without charge sources (free field - free photon)

$$\partial_\nu \partial^\nu A_\mu = 0 \tag{9}$$

The solution for free field is:

$$A^\mu(x) = \varepsilon^\mu(\vec{q}) \cdot \exp(-iqx) \tag{10}$$

where ε is 4-vector of polarization a q *4*-momentum carried by field (photon).

The important moments:

- For the solution (10) from Eq. (9) follows: $q^2 = 0$ - this corresponds to zero mass of photons.
- Lorentz condition ($q_\mu \varepsilon^\mu = 0$) and additional freedom (8) enable to choose potential A^μ, in such a way that the vector polarization reads: $\varepsilon_0 \equiv 0$ and hence:

$$\vec{\varepsilon} \cdot \vec{q} = 0 \tag{11}$$

This means that electromagnetic field is polarized transversally and the vector of polarization has two independent components, i.e. the base of vector of polarization contains 2 elements: $\vec{\varepsilon}^{(\lambda)}, \lambda = 1,2$.

In general the free electromagnetic field is described by the potential:

$$\vec{A}(x) = \int \frac{d\vec{q}}{(2\pi)^{3/2}\sqrt{2\omega}} \sum_{\lambda=1}^{2} \vec{\varepsilon}^{(\lambda)}(\vec{q}) \cdot \left(a_\lambda^{(-)}(\vec{q}) e^{-iqx} + a_\lambda^{(+)}(\vec{q}) e^{iqx} \right) \tag{12}$$

where $q = (\omega, \vec{q})$ and two terms in (…) correspond to solution with positive and negative energy (frequency): $q^0 = \pm\sqrt{\vec{q}^2} = \pm\omega$ (hence ω is defined positively).

Energy electromagnetic field

$$H = \frac{1}{2}\int d^3\vec{x}\cdot\left(\vec{E}^2+\vec{B}^2\right) = \frac{1}{2}\int d^3\vec{x}\cdot\left[\left(\partial_t\vec{A}\right)^2+\left(\nabla\times\vec{A}\right)^2\right] =$$
$$= \sum_\lambda \int \frac{d^3\vec{q}}{(2\pi)^3}\cdot\omega\cdot\frac{1}{2}\underbrace{\left(\frac{a_\lambda^{(-)}(\vec{q})a_\lambda^{(+)}(\vec{q})+a_\lambda^{(+)}(\vec{q})a_\lambda^{(-)}(\vec{q})}{2\omega}\right)}_{*} \qquad (14)$$

where $\underbrace{\cdots}_{*}$ represents density of photons (waves) with energy (frequency) $\omega = |\vec{q}|$ in the system of electromagnetic field.

Conclusion. Electromagnetic field is possible to interpret in two ways:

- through $\vec{E}$ and $\vec{B}$ (intensity of electric field and magnetic field),
- as a system of photons (quanta of field).