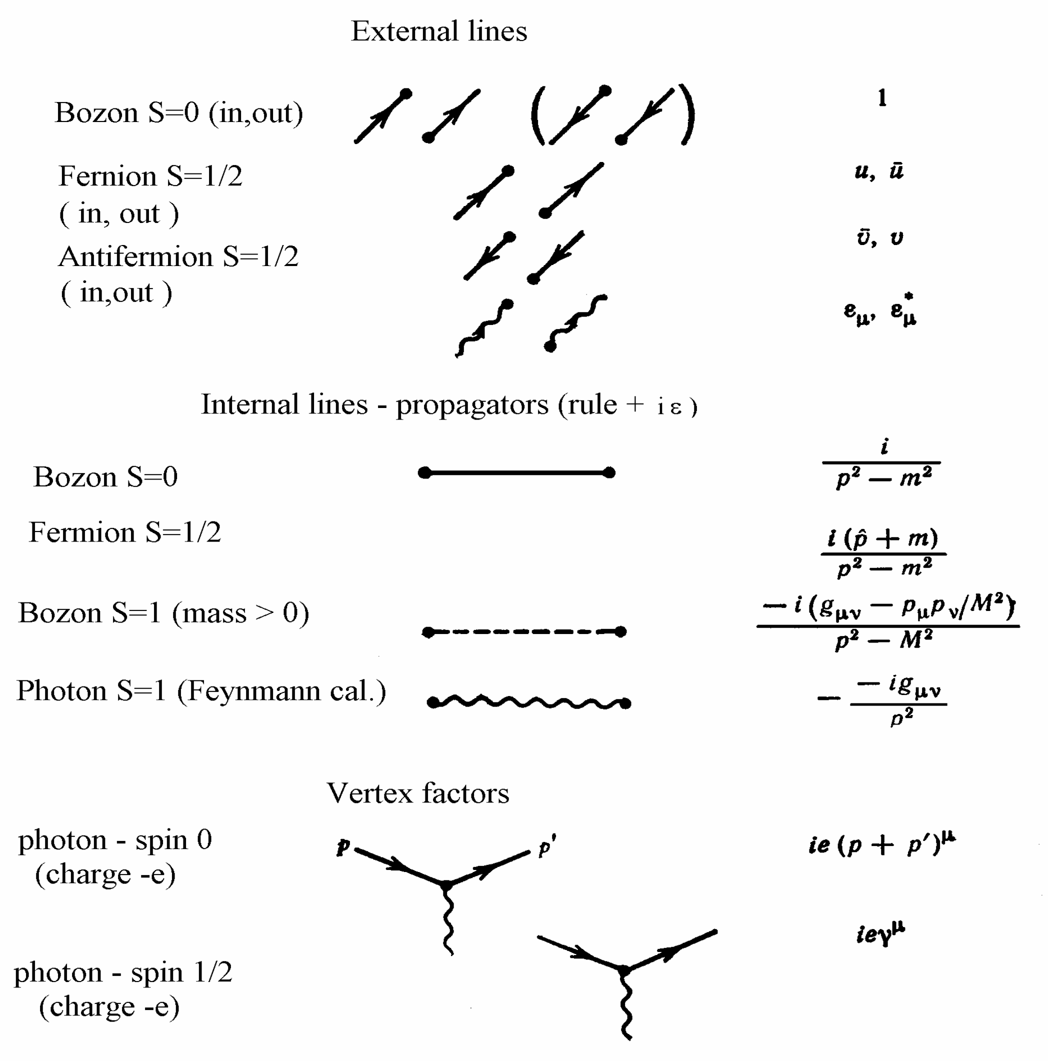
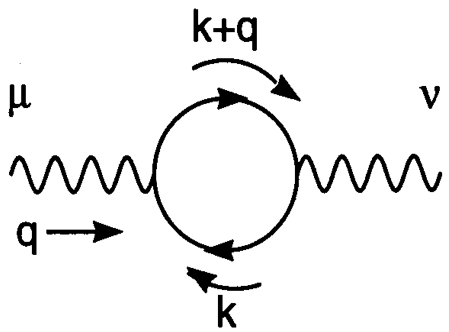
1. Feynman rules

The basic rules for creating matrix elements using the Feynman diagrams are summarized below.



Loops in diagrams



In case of fermion loop it is needed:

- Add the factor (–1),
- Make a trace of corresponding γ–matrices,
- a integrate through momentum circulating in loop ⇒

$$F.S. = (-1)\int d^4k \ (ie\gamma^\mu)_{\alpha\beta} \cdot \frac{i(\hat{k}+m)_{\beta\lambda}}{k^2-m^2} \cdot (ie\gamma^\nu)_{\lambda\delta} \cdot \frac{i(\hat{q}+\hat{k}+m)_{\delta\alpha}}{(q+k)^2-m^2}$$

$$= (-1)i^2(ie)^2\int d^4k \cdot \frac{Tr\left(\gamma^\mu(\hat{k}+m)\gamma^\nu(\hat{q}+\hat{k}+m)\right)}{(k^2-m^2)(q+k)^2-m^2} \qquad (1.1)$$

where

$$\hat{k} = k_\mu\gamma^\mu, \qquad \hat{q} = q_\mu\gamma^\mu$$

The repeating indexes are the summing indexes.

Remark. The momentum in loop is limited only by the law of momentum conservation. As the relation $q = (q-k)+k$ is valid for all k, we must integrate through all possible k.

The relation of completeness

Let us consider a particle with ***spin 1***, mass ***m*** and vector of polarization ε^μ for the spin sum is valid:

$$\sum_s \varepsilon^\mu\varepsilon^{*\nu} = -g^{\mu\nu} + q^\mu q^\nu / m^2 \qquad (1.2)$$

2. On quantization of field

Let us assume that the content of a physical system is a field $\Psi(x)$ or of system fields. Then application of the variation principle on the action of the system ($S = \int L dt, \quad \delta S = 0$, $L \equiv$ the system Lagrangian) leads to the Lagrange equations:

$$\partial_{\mu}\left(\frac{\partial \Lambda}{\partial(\partial_{\mu}\Psi)}\right) - \frac{\partial \Lambda}{\partial \Psi} = 0 \tag{2.1}$$

Where Λ is the Lagrangian density.

The basic fields with which we have to do in the elementary particle physics are:

- Complex scalar field:

$$\Lambda = \partial_{\mu}\varphi^{*} \cdot \partial^{\mu}\varphi - m^{2}\varphi^{*}\varphi , \tag{2.2}$$

The corresponding LR:

$$\left(\partial_{\mu}\partial^{\mu} + m^{2}\right)\varphi = 0, \quad \left(\partial_{\mu}\partial^{\mu} + m^{2}\right)\varphi^{*} = 0 \tag{2.3}$$

- The spinor (Dirac) field:

$$\Lambda = \frac{i}{2}\left[\overline{\Psi}\gamma^{\mu} \cdot \partial_{\mu}\Psi - \left(\partial_{\mu}\overline{\Psi}\right)\gamma^{\mu}\Psi\right] - m\,\overline{\Psi}\Psi \ , \tag{2.4}$$

The corresponding LR:

$$\left(i\gamma_{\mu}\partial^{\mu} - m\right)\Psi = 0, \qquad \overline{\Psi}\left(i\gamma_{\mu}\overleftarrow{\partial}^{\mu} + m\right) = 0 \tag{2.5}$$

- The electromagnetic field with external source:

$$\Lambda = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_{\mu}j^{\mu}, \tag{2.6}$$

The corresponding LR:

$$\partial_{\nu}\partial^{\nu}A^{\mu} - \partial_{\mu}\left(\partial^{\nu}A_{\nu}\right) = \quad j^{\mu} \tag{2.7}$$

The Lagrange equation for:

- Scalar field (≡ KG equation) describes the particles with spin 0, i.e. the quanta of scalar field,
- Spinor field (≡ Dirac equation) describes the particles with spin 1/2, i.e. the quanta of spinor field,
- Electromagnetic field (≡ Maxwell equation) describes photons (spin 1, $m=0$), i.e. the quanta of electromagnetic field.

Scalar and spinor field do not have classical interpretation – only the quantum one, while the electromagnetic field has both the classical interpretation (intensity of electric and magnetic field), as well as the quantum interpretation – a system photon.

The above mentioned Lagrange equations (KGR, DR, MR) are equations of classical fields. In application to one particle they give the quantum mechanics of:

- boson with spin 0 (KGR)
- fermion with spin 1 (DR)
- photon (spin 1, m=0) (MR).

From comparison of the structure of the Lagrange equation of system of fields (1) with that of classical system with N degrees of freedom

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=0, \quad i=1\ldots N\,, \tag{2.8}$$

it follows:

$$q \rightarrow \Psi \quad \text{a} \quad t \rightarrow x^{\mu} \tag{2.9}$$

- the generalized coordinate of the field system is $\Psi(x)$;
- the generalized momentum of the field system is $\pi(x)=\dfrac{\partial \Lambda}{\partial\left(\partial_{\mu}\Psi\right)}$.

***Quantum Mechanics (QM)*:** **the quantization consists in the replacement of physical variables by corresponding quantum operators:**

$$q \rightarrow \hat{q} = q \quad \text{a} \quad p \rightarrow \hat{p} = -i\hbar\nabla \tag{2.10}$$

where in general the quantum operators do not commute:

$$[\hat{q}, \hat{p}] = i\hbar \tag{2.11}$$

The operators of quantum variables act in a Hilbert space that represents the quantum states of a physical system.

***Quantum Field Theory (QFT)*:** **for field quantization is valid an analogical scheme as in QM:**

$$\Psi(x) \rightarrow \hat{\Psi}(x) \quad \text{a} \quad \pi(\mathrm{x}) \rightarrow \hat{\pi}(x), \tag{2.12}$$

where $\hat{\Psi}(x)$ and $\hat{\pi}(x)$ are the operators satisfy the following commutation relations:

$$\begin{aligned} \left[\hat{\Psi}(\vec{x},t),\hat{\pi}(\vec{x}',t)\right] &= i\delta(\vec{x}-\vec{x}') \\ \left[\hat{\Psi}(\vec{x},t),\hat{\Psi}(\vec{x}',t)\right] &= 0 \\ \left[\hat{\pi}(\vec{x},t),\hat{\pi}(\vec{x}',t)\right] &= 0 \end{aligned} \tag{2.13}$$

The $\hat{\Psi}$ -operators could be expressed by means of annihilation and creation operators. Using $\hat{\Psi}$ -operators or annihilation and creation operators one can construct Hamiltonian of the field system and this Hamiltonian will act in the corresponding Fok space (see further).

The charge scalar field quantization.

If our physical system is made up of a complex scalar field then the density of its Lagrangian is

$$L = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \tag{2.14}$$

Where ϕ and ϕ* can be treated as independent fields and two canonically conjugate to them fields are:

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}^* , \quad \pi^* = \frac{\partial L}{\partial \dot{\phi}^*} = \dot{\phi} \tag{2.15}$$

Finally it leads to the Hamiltonian:

$$H = \int d^3x \left(\pi \partial_0 \phi + \pi^* \partial_0 \phi^* - L \right) = \int d^3x \left(\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \right) \tag{2.16}$$

Quantization of the field is achieved by replacement of the fields ϕ, ϕ*, π, π* by the field operators: $\hat{\phi}, \hat{\phi}^\dagger, \hat{\pi}, \hat{\pi}^\dagger$ which are required to fulfill at the same time the commutation relations:

$$\left[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{x}',t) \right] = \left[\hat{\phi}^\dagger(\vec{x},t), \hat{\pi}^\dagger(\vec{x}',t) \right] = i\delta(\vec{x} - \vec{x}') \tag{2.17}$$

The Lagrange equations for the complex scalar field are Klein-Gordon equations (KGE). In general the solution of KGE is a superposition of the plane waves:

$$\phi(x) = \int \frac{d^3 \vec{p}}{(2\pi)^{3/2} \sqrt{2E}} \left[a^{(-)}(\vec{p}) e^{-ipx} + a^{(+)}(\vec{p}) e^{ipx} \right], \qquad \begin{matrix} x \equiv \left(x^0, \vec{x} \right) \\ p \equiv \left(E, \vec{p} \right) \end{matrix} \tag{2.18}$$

At the field quantization the coefficients $a^{(-)}(\vec{p})$ and $a^{(+)}(\vec{p})$ will be replaced by the operators $\hat{a}_{\vec{p}}$ and $\hat{b}^\dagger_{\vec{p}}$ and in the field ϕ* the coefficients $\left(a^{(-)}(\vec{p})\right)^*$ and $\left(a^{(+)}(\vec{p})\right)^*$ by the operators $\hat{a}^\dagger_{\vec{p}}$ and $\hat{b}_{\vec{p}}$. It can be easily shown that to fulfill the commutation relation (2.17) the operators $\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}}, \hat{b}_{\vec{p}}$ and $\hat{b}^\dagger_{\vec{p}}$ should obey:

$$\left[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'} \right] = \left[\hat{b}_{\vec{p}}, \hat{b}^\dagger_{\vec{p}'} \right] = i\delta(\vec{p} - \vec{p}') \tag{2.19}$$

All others commutators are equal to 0. The $\hat{a}_{\vec{p}}$, $\hat{b}_{\vec{p}}$ are called annihilation operators and the $\hat{a}^\dagger_{\vec{p}}$, $\hat{b}^\dagger_{\vec{p}}$ creation operators of particle and antiparticle, respectively. Using the annihilation and creation operators we can express the Hamiltonian (2.16) through them. The fact that $\hat{a}_{\vec{p}}, \hat{a}^{\dagger}_{\vec{p}}$, $\hat{b}_{\vec{p}}$ and $\hat{b}^{\dagger}_{\vec{p}}$ are operators, and in general do not commute, means that the order of these operators is important (while it was not the case for the corresponding classical quantities). Starting from the Hamiltonian (2.16) replacing the classical fields by the operator ones and the latter expressing through the annihilation and creation operators we get:

$$\hat{H} = \int d^3\vec{p}\,\omega_{\vec{p}}\left(\hat{a}_{\vec{p}}\hat{a}^{\dagger}_{\vec{p}} + \hat{b}^{\dagger}_{\vec{p}}\hat{b}_{\vec{p}}\right) = \int d^3\vec{p}\,\omega_{\vec{p}}\left(\hat{a}^{\dagger}_{\vec{p}}\hat{a}_{\vec{p}} + \hat{b}^{\dagger}_{\vec{p}}\hat{b}_{\vec{p}} + 1\right) \tag{2.20}$$

The fact that in (2.20) after commutation we get 1 under the integral is unpleasant as after the integration we get infinity (infinite energy of vacuum). But in theory this 1 under the integral is ignored as the so called normally ordered Hamiltonian is used as Hamiltonian of physical system:

$$\hat{H} =: \int d^3\vec{p}\,\omega_{\vec{p}}\left(\hat{a}_{\vec{p}}\hat{a}^{\dagger}_{\vec{p}} + \hat{b}^{\dagger}_{\vec{p}}\hat{b}_{\vec{p}}\right): \; = \int d^3\vec{p}\,\omega_{\vec{p}}\left(\hat{a}^{\dagger}_{\vec{p}}\hat{a}_{\vec{p}} + \hat{b}^{\dagger}_{\vec{p}}\hat{b}_{\vec{p}}\right) \tag{2.21}$$

Where : -- : is the symbol of normal order (all creation operators are left to annihilation operators) and $\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$.

As it will be shown later the Hamiltonian $\hat{H}$ acts in Fok space and the operator $\hat{a}^{\dagger}_{\vec{p}}\hat{a}_{\vec{p}}$ is an operator of number of particles with momentum $\vec{p}$.

The spinor field quantization

In case of the spinor field the full solutions of Dirac equations are:

$$\begin{aligned} \Psi(x) &= \int \frac{d\vec{p}}{(2\pi)^{3/2}\sqrt{2E}} \sum_{\lambda}\left(a^{(+)}_{\lambda}(\vec{p})\cdot u_{\lambda}(\vec{p})\cdot e^{-ipx} + a^{(-)}_{\lambda}(\vec{p})\cdot v_{\lambda}(\vec{p})\cdot e^{ipx}\right) \\ \bar{\Psi}(x) &= \int \frac{d\vec{p}}{(2\pi)^{3/2}\sqrt{2E}} \sum_{\lambda}\left((a^{(+)}_{\lambda}(\vec{p}))^{*}\cdot \bar{u}_{\lambda}(\vec{p})\cdot e^{ipx} + (a^{(=)}_{\lambda}(\vec{p}))^{*}\cdot v_{\lambda}(\vec{p})\cdot e^{-ipx}\right) \end{aligned} \tag{2.22}$$

The Lagrangian of spinor field:

$$L = \frac{i}{2}\left[\bar{\Psi}\gamma^{\mu}\left(\partial_{\mu}\Psi\right) - \partial_{\mu}\bar{\Psi}\gamma^{\mu}\Psi\right] - m\bar{\Psi}\Psi \tag{2.23}$$

The canonical momentum:

$$\pi(x) = \frac{\partial L}{\partial\dot{\Psi}(x)} = i\Psi^{+}(x) \tag{2.24}$$

The Hamiltonian:

$$H = \int d^3x \cdot \pi \cdot \dot{\Psi} - L = \int d^3x \cdot \Psi^+ i \frac{\partial}{\partial t} \Psi \qquad (2.25)$$

In general the field 4-momentum reads:

$$P_\mu = i \int d^3x \cdot \overline{\Psi} \gamma^0 \partial_\mu \Psi \qquad (2.26)$$

And for its charge one gets:

$$Q = i \int d^3x \cdot \overline{\Psi} \gamma^0 \Psi \qquad (2.27)$$

The field Hamiltonian after having the field function explicitly expressed is:

$$H = \int d^3p \cdot E \cdot \sum_\lambda \left[(a_\lambda^{(+)}(\vec{p}))^* a_\lambda^{(+)}(\vec{p}) - (a_\lambda^{(-)}(\vec{p}))^* a_\lambda^{(-)}(\vec{p}) \right] \qquad (2.28)$$

The field charge after the explicit expression of the field function:

$$Q = \int d^3p \cdot \sum_\lambda \left[(a_\lambda^{(+)}(\vec{p}))^* a_\lambda^{(+)}(\vec{p}) + (a_\lambda^{(-)}(\vec{p}))^* a_\lambda^{(-)}(\vec{p}) \right] \qquad (2.29)$$

Now let us assign operators to the coefficients $a_\lambda^{(\pm)}$ and $\left(a_\lambda^{(\pm)}\right)^*$:

$$a_\lambda^{(+)} \to a_\lambda \quad \left(a_\lambda^{(+)}\right)^* \to a_\lambda^+ , \qquad a_\lambda^{(-)} \to b_\lambda^+ \quad \left(a_\lambda^{(-)}\right)^* \to b_\lambda \qquad (2.30)$$

where a_λ $\left(a_\lambda^+\right)$ are the annihilation (creation) operators for electrons

b_λ $\left(b_\lambda^+\right)$ are the annihilation (creation) operators for positrons.

To have the energy of the system positively defined and on the other side its charge should have a possibility to be positive or negative, the following anti-commutation relations should be fulfilled:

$$\left\{ a_\lambda(\vec{k}),\ a_{\lambda'}^+(\vec{k}') \right\} = \delta_{\lambda\lambda'} \cdot \delta^3(\vec{k} - \vec{k}') \qquad \left\{ b_\lambda(\vec{k}),\ b_{\lambda'}^+(\vec{k}') \right\} = \delta_{\lambda\lambda'} \cdot \delta^3(\vec{k} - \vec{k}') \qquad (2.31)$$

On quantization of the electromagnetic field

For the vector potential of electromagnetic field one can write:

$$\vec{A}(x) = \int \frac{d\vec{k}}{(2\pi)^{3/2} \sqrt{2\omega}} \sum_\lambda \vec{\varepsilon}^{(\lambda)}(\vec{k}) \cdot \left(a_\lambda^{(-)}(\vec{k}) \cdot e^{-ikx} + a_\lambda^{(+)}(\vec{k}) \cdot e^{ikx} \right) \qquad (2.32)$$

On the base of the vector potential (2.32) the vectors of the electric intensity $\vec{E}$ and magnetic induction $\vec{B}$ can be easily found using the relations: $\vec{E} = -\partial_t \vec{A} - \Delta A_0$ and $\Delta \times \vec{B} = (1/c)\partial_t \vec{E}$. The quantities $\vec{E}$ and $\vec{B}$ represent classical interpretation of electromagnetic field. To find the quantum representations of this field we will use (2.32) to express the energy of electromagnetic field:

$$H_{em} = \sum_{\lambda} \int \frac{d\vec{k}}{(2\pi)^3\, 2\omega} \cdot \omega \cdot \frac{1}{2}\left(a_{\lambda}^{(-)}(\vec{k}) \cdot a_{\lambda}^{(+)}(\vec{k}) + a_{\lambda}^{(+)}(\vec{k}) \cdot a_{\lambda}^{(-)}(\vec{k}) \right) \tag{2.33}$$

Where $k = (\omega, \vec{k})$ and $\omega = \sqrt{\vec{k}^2}$

Remark. The Hamiltonian (2.33) one can write: $H_{em} = \sum_{\lambda} \int \frac{d\vec{k}}{(2\pi)^3\, 2\omega} \cdot \omega \cdot a_{\lambda}^{(-)}(\vec{k}) \cdot a_{\lambda}^{(+)}(\vec{k})$ *however we let it be in the form shown above due to the reasons needed for its quantization.*

The field quantization in the replacement of the coefficients $a_{\lambda}^{(\pm)}$ by the operators:

$$\begin{aligned} \vec{A}(x) &\to \hat{\vec{A}}(x) && (\Psi\text{–operator}) \\ a_{\lambda}^{(-)}(\vec{k}) &\to \hat{a}_{\lambda}(\vec{k}) && (\text{annihilation operator}) \\ a_{\lambda}^{(+)}(\vec{k}) &\to \hat{a}_{\lambda}^{+}(\vec{k}) && (\text{creation operator}) \end{aligned} \tag{2.34}$$

To have the Hamiltonian (2.15) positively defined the commutation relations for the creation and annihilation operators should obey:

$$\left[\hat{a}_{\lambda}(\vec{k}),\ \hat{a}_{\lambda}^{+}(\vec{k}') \right] = \delta\left(\vec{k} - \vec{k}'\right) \cdot \delta_{\lambda\lambda'} \tag{2.35}$$

The Fok space

Using the operators $\hat{\Psi}$ a $\hat{\pi}$ (or creation and annihilation operators, see further) it is possible to construct the Hamiltonian H of a system of interacting fields. The operator H acts in Fok space that contains the states:

$$\Phi \equiv \{|\phi\rangle\} \quad \text{where} \quad |\phi\rangle = \underbrace{c_0|\phi_0\rangle}_{\text{vacuum}} + \underbrace{c_1|\phi_1\rangle}_{1\text{–particle state}} + \underbrace{c_2|\phi_2\rangle}_{2\text{–particle state}} + \cdots \tag{2.36}$$

The creation and annihilation operators

The operators of creation and annihilation $\boldsymbol{a}(\boldsymbol{k}), \boldsymbol{a}^{+}(\boldsymbol{k})$ play a key role in multi-particle interpretation of quantum field theory. Let us begin with the operator

$$\boldsymbol{N}(\boldsymbol{k}) = \boldsymbol{a}^{+}(\boldsymbol{k})\boldsymbol{a}(\boldsymbol{k}) \tag{2.37}$$

Its eigen values we denote as ***n(k)***:

$$\boldsymbol{N}(\boldsymbol{k})|\boldsymbol{n}(\boldsymbol{k})\rangle = \boldsymbol{n}(\boldsymbol{k})|\boldsymbol{n}(\boldsymbol{k})\rangle \tag{2.38}$$

Using the commutation relations for $\boldsymbol{a}(\boldsymbol{k}), \boldsymbol{a}^{+}(\boldsymbol{k})$ we get:

$$\begin{aligned} \left[\boldsymbol{N}(\boldsymbol{k}), \boldsymbol{a}^{+}(\boldsymbol{k})\right] &= \boldsymbol{a}^{+}(\boldsymbol{k}) \\ \left[\boldsymbol{N}(\boldsymbol{k}), \boldsymbol{a}(\boldsymbol{k})\right] &= -\boldsymbol{a}(\boldsymbol{k}) \end{aligned} \tag{2.39}$$

From where it follows:

$$\begin{aligned} \boldsymbol{N}(\boldsymbol{k})\boldsymbol{a}^{+}(\boldsymbol{k})|\boldsymbol{n}(\boldsymbol{k})\rangle &= \boldsymbol{a}^{+}(\boldsymbol{k})\boldsymbol{N}(\boldsymbol{k})|\boldsymbol{n}(\boldsymbol{k})\rangle + \boldsymbol{a}^{+}(\boldsymbol{k})|\boldsymbol{n}(\boldsymbol{k})\rangle \\ &= (\boldsymbol{n}(\boldsymbol{k})+1)\boldsymbol{a}^{+}(\boldsymbol{k})|\boldsymbol{n}(\boldsymbol{k})\rangle \end{aligned} \tag{2.40}$$

Hence if the state $|\boldsymbol{n}(\boldsymbol{k})\rangle$ is an eigenstate of operator ***N(k)*** with the eigenvalue ***n(k)***, then $\boldsymbol{a}^{+}(\boldsymbol{k})|\boldsymbol{n}(\boldsymbol{k})\rangle$ is the eigenstate of operator ***N(k)*** with the eigenvalue ***n(k)+1***.

In an analogical way it can be shown that $\boldsymbol{a}(\boldsymbol{k})|\boldsymbol{n}(\boldsymbol{k})\rangle$ is the eigenstate of operator ***N(k)*** with the eigenvalue ***n(k)-1***.

The scheme of interaction in quantum field theory (QFT)

Initial state of field system	Interaction	Final state of field system
(system of free particles)	——————————→	(system of free particles)
	interaction Hamiltonian	

	probability	
$e^{+}(\vec{k}_1)\, e^{-}(\vec{k}_2)$	——————————→	$\mu^{+}(\vec{p}_1)\, \mu^{-}(\vec{p}_2)$
	of transition	

The goal of QFT is to find the element $\left\langle \boldsymbol{\phi}_{in} \middle| \boldsymbol{H} \middle| \boldsymbol{\phi}_{out} \right\rangle$ that represents probability of transition from the state $\left| \boldsymbol{\phi}_{in} \right\rangle$ to the state $\left| \boldsymbol{\phi}_{out} \right\rangle$ due to interaction. The QFT gives a recipe for calculation of the transition amplitude (matrix element). The practical realization of this recipe present the technique of Feynmann diagrams.