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On propagators

Propagator describes transport of a virtual particle i.e. a particle that is not on its mass shell $p^2 \neq m^2$. In diagrams to virtual particles respond internal lines (see Fig.1).

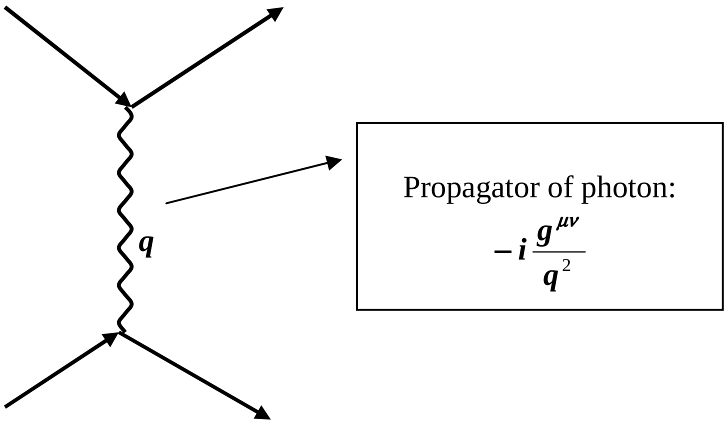

Fig. 1: Propagator of photon (momentum *q*)

To be a virtual particle can be not only photon but also any other particle. As an example the Compton effect can be brought (Fig. 2).

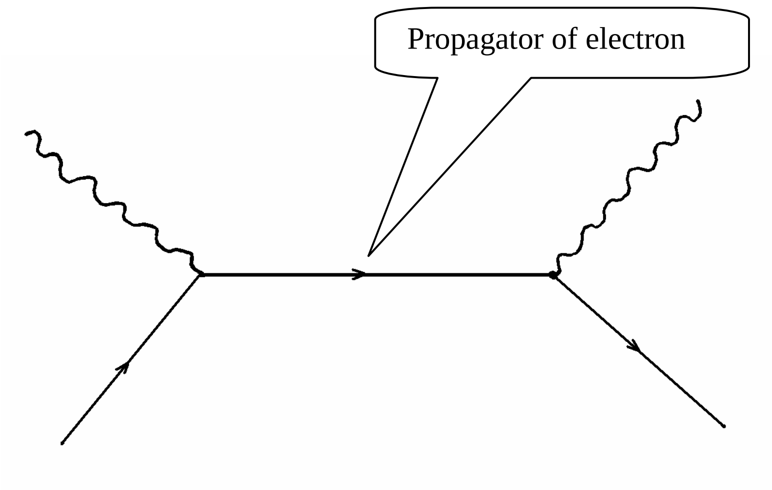

Fig. 2: Compton effect - propagator electron.

How to find propagator?

There is a very simple recipe – it is an inverse operator to the operator of equation of motion for a given particle (Lagrange equation of a field – a given particle is a quantum of this field) in ***p***-representation (space of momentum) we will apply this recipe on particles with different spin.

Propagator particle with spin 0.

The equation of motion for particle with ***spin 0*** moving in the field of potential ***V*** is:

$$i\left(\partial_\mu \partial^\mu + m^2\right)\varphi = -iV\varphi$$

As the operator of 4-momentum is $p^\mu = -i\partial^\mu$ for the propagator we get:

$$\frac{1}{i\left(-p^2+m^2\right)} = \frac{i}{p^2-m^2} \qquad (1)$$

Propagator of particle with spin ½.

The equation of motion for particle with spin ½ moving in the field of potential A_μ reads: $-i\left(p^\mu \gamma_\mu - m\right)\psi = ie\gamma_\mu A^\mu$

Hence the propagator is:

$$\frac{1}{-i\left(\hat{p}-m\right)} = \frac{i\left(\hat{p}+m\right)}{p^2-m^2} \qquad \hat{p} \equiv p^\mu \gamma_\mu \qquad (2)$$

Propagator of photon (spin 1 a m=0).

The equation of motion for photon is:

$$\left(g^{\mu\nu}\partial_\rho \partial^\rho - \partial^\mu \partial^\nu\right)A_\nu = j^\mu$$

The first of all we need to get rid of the calibration freedom in A_ν. As to the operator $g^{\mu\nu}\partial_\rho\partial^\rho - \partial^\mu\partial^\nu$ there does not exist an inverse operator: application of Lorentz condition: $\partial_\mu A^\mu = 0$ leads to: $g^{\mu\nu}\partial_\rho\partial^\rho A_\nu = j^\mu$.

This equation leads to the propagator (***q*** is photon 4-momentum):

$$i\frac{-g_{\mu\nu}}{q^2} \qquad (3)$$

With the propagator of photon we have already been acquainted with.

Propagator of particle with spin 1 and m≠0.

The equation of motion for a free particle with $S=1$ and $m\neq 0$ we will get from the equation for photon by replacing: $\partial_\mu \partial^\mu \to \partial_\mu \partial^\mu + m^2$, hence as a result we have:

$$\left(g^{\mu\nu}\left(\partial_\rho \partial^\rho + m^2\right) - \partial^\mu \partial^\nu\right) B_\nu = 0$$

From there the propagator reads: $i\left(g^{\mu\nu}\left(-p^2+m^2\right)+p^\mu p^\nu\right)^{-1} = \dfrac{i\left(-g^{\mu\nu}+\dfrac{p^\mu p^\nu}{m^2}\right)}{p^2-m^2}$ (4)

Green function and propagator

Definition. Green function is a response to point-like source. Let us consider a motion of electron in an electromagnetic magnetic field:

$$\left(i\gamma_\mu \partial^\mu - m\right)\psi = -e\gamma_\mu A^\mu \tag{5}$$

The right side of Eq. 5 can be considered as a source function of electron-positron field. The problem of electron motion in field A_μ it is possible to solve in such a way that first we will solve:

$$\left(i\gamma_\mu \partial^\mu - m\right) G_F(x-x') = \delta^4(x-x') \tag{6}$$

Where $G_F(x-x')$ is a response (wave) in point x from point-like unit source in the position x'. The solution of Eq.5 can be then expressed as follows:

$$\psi(x) = -e\int d^4x' G_F(x-x')\cdot \gamma_\mu A^\mu(x') \tag{7}$$

The Green function G_F we look for in the form of Fourier expansion:

$$G_F(x-x') = \frac{1}{(2\pi)^4}\int d^4p \cdot S_F(p) e^{-ip(x-x')} \tag{8}$$

After putting it into (6) and having expanded the δ-function into Fourier series ($\delta^4(x-x') = \dfrac{1}{(2\pi)^4}\int d^4p \cdot e^{-ip(x-x')}$) one obtains:

$$\left(p^\mu \gamma_\mu - m\right) S_F(p) = 1 \quad \Rightarrow \quad S_F(p) = \frac{1}{\hat{p}-m} = \frac{\hat{p}+m}{p^2-m^2} \tag{9}$$

For finding the G_F one must know how to treat the singularities in the $S_F(p)$:

$$p^2 - m^2 = p_0^2 - \left(\vec{p}^2 + m^2\right) = (p_0+E)(p_0-E)$$

where $E=\sqrt{\vec{p}^2+m^2}$. Hence $S_F(p)$ has the poles in $p_0=\pm E$.

How to find a general solution of the Green function equation?

From the mathematical analysis it is known: ***the general solution of Eq. 6 ≡***

≡particular solution of (6) + general solution of homogeneous Eq. (6)

(Eq. 6 without right side)

The solution of homogeneous Eq. 6 is known as it is the Dirac equation. Let us denote the solution with positive energy as $D^{-}(x-x')$ and that with negative energy as $D^{+}(x-x')$.

Then in general the Green function (solution of Eq. 6) reads:

$$G_F(x-x')=G_{part}(x-x')+c_1 D^{-}(x-x')+c_2 D^{+}(x-x') \tag{10}$$

The coefficients c_1 and c_2 are fixed by boundary conditions.

Boundary conditions

Let us assume that a unit source of spinor field is in the point $x'\equiv(t',\vec{x}')$ and we want the Green function to describe:

1. propagation of electron with positive energy ($E>0$) in time forward ($t>t'$) (***OP1***)
2. propagation of electron with negative energy ($E<0$) in time backward ($t<t'$) (***OP2***)

The physical interpretation.

⇒ the boundary condition ***OP1*** means:

If in the moment t' in the position $\vec{x}'$ appeared an ***electron***, then the G_F will give probability of its appearance in the position $\vec{x}$ in some later moment $t\ (>t')$.

⇒ the boundary condition ***OP2*** means:

If in the moment t in the position $\vec{x}$ appeared a ***positron***, then the G_F will give a probability of its appearance in the position $\vec{x}'$ in some later moment $t'\ (>t)$.

We will show that if the condition ***OP1*** is fulfilled the Green function describes propagation of electron forward in time and if the condition ***OP2*** is fulfilled, it describes propagation of positron in time forward.

Let us express the G_F *via* its Fourier pictures in the momentum space, see (7) and (8):

$$G_F(x-x') = \frac{1}{(2\pi)^4}\int d^3\vec{p} \;\cdot\; e^{i\vec{p}(\vec{x}-\vec{x}')}\int dp_0 \frac{(\hat{p}+m)e^{-ip_0(t-t')}}{(p_0-E)(p_0+E)} \tag{11}$$

The integrand $S_F(p)$ in $\int dp_0 \cdots$ does have the poles for $p_0 = \pm E$.

The integral $I(p_0) = \int dp_0 \cdots$ is calculated using the method of complex variable that consists in the replacement:

$$\int_{-\infty}^{\infty} dp_0 \cdots \rightarrow \int_C dp_0 \cdots \tag{12}$$

Where the C is an integration contour in the p_0 complex plane containing the interval (–∞, ∞) and circumnavigating the poles $p_0 = \pm E$ in such a way that to fulfill boundary conditions ***OP1*** and ***OP2.***

Let us require that the Green function describes propagation of electron with positive energy for $t > t'$(the condition ***OP1***). We will show that fulfillment of the ***OP1*** requires the poles to be circumvented as is shown in Fig. 3 –elusion of the pole $p_0 = +E$.

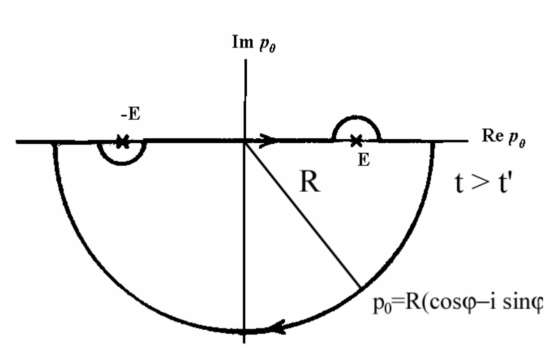

Fig. 3: The integration contour for t > t′, it is enclosed in the lower half-plane (pole $p_0 = +E$)

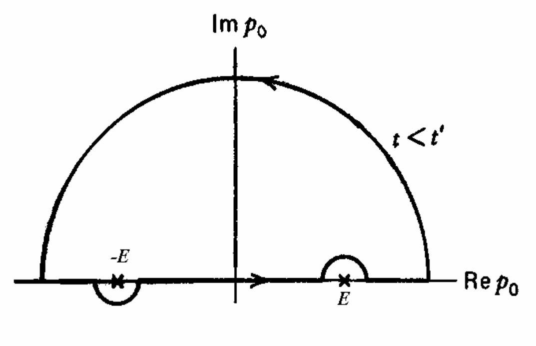

Fig. 4: The integration contour for t<t′; it is enclosed in the upper half-plane (pole $p_0 = -E$)

Using the Cauchy theorem for the integral on the contour ***C*** (Fig. 3) we obtain:

$$I(p_0)=\int_C dp_0 \frac{(\hat{p}+m)\ e^{-ip_0(t-t')}}{(p_0-E)(p_0+E)}=\int_P dp_0\cdots+\int_K dp_0\cdots=-2\pi i\ \frac{(\hat{p}+m)\ e^{-iE(t-t')}}{2E} \tag{13}$$

Our goal is to make the integral $\int_K dp_0 \cdots$ on half circumference K to converge to 0 for $R\to\infty$.

From the convergence view point the critical factor is $e^{ip_0(t-t')}$. However for $t>t'$ and any point on lower half-circumference $p_0 = R(\cos\varphi - i\sin\varphi)$ is valid:

$$e^{-ip_0(t-t')}=e^{-iR\cos\varphi(t-t')}\cdot e^{-R\sin\varphi(t-t')}\xrightarrow[R\to\infty\ \wedge\ t>t']{} 0 \tag{14}$$

Conclusion. In the case when the contour C is enclosed in the lower half-plane for $t>t'$, it leads to:

$$\int_K dp_0\cdots\xrightarrow[R\to\infty]{}0\ ,\ \Rightarrow\ \int_C dp_0\cdots=\int_P dp_0\cdots \tag{15}$$

If now we shift the poles: $p_0=\pm E\ \to\ p_0=\pm\left(E-i\varepsilon\right)$ and replace the integration contour C by $C'\equiv(-\infty,\infty)\cup K$ (see Fig. 5) then we will get the following:

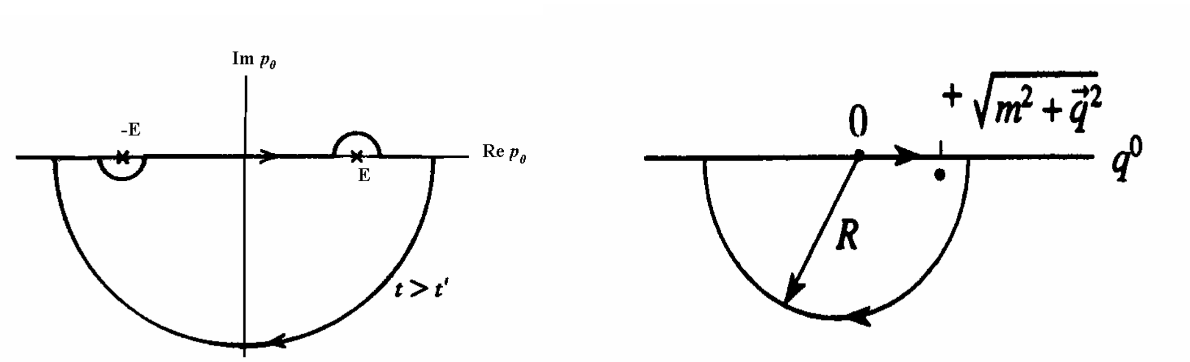

Fig. 5: The replacement of the integration contour C (see the left plot) by the contour C′ (see the right plot).

$$\begin{aligned}\int_{-\infty}^{\infty} dp_0\cdots &=\lim_{\varepsilon\to 0}\int_{C'} dp_0 \frac{(\hat{p}+m)\exp[-ip_0(t-t')]}{(p_0-E+i\varepsilon)(p_0+E-i\varepsilon)}\\ &=\lim_{\varepsilon\to 0}\left(-2\pi i\frac{(\hat{p}+m)\exp[-iE(t-t')-\varepsilon(t-t')]}{2(E-i\varepsilon)}\right)=-2\pi i\frac{(\hat{p}+m)\exp[-iE(t-t')]}{2E}\end{aligned} \tag{16}$$

Putting into Eq. 11 instead of the integral $I(p_0)=\int dp_0\cdots$ the calculated quantity (16), the Green function in case of $t>t'$ reads:

$$G_F(x-x')=\frac{-i}{(2\pi)^3}\int d\vec{p}\,\frac{(\hat{p}+m)\exp[-ip(x-x')]}{2E} \tag{17}$$

where $p\equiv(E,\vec{p})$, $E=\sqrt{\vec{p}^{\,2}+m^2}$ a $\hat{p}+m=\sum_{s=1,2}u^{(s)}(p)\bar{u}^{(s)}(p)$ is a projection operator making projection into the state of electron with positive energy E (see remark), thereby the G_F describes propagation of electron forward in time.

Remark. Let us assume that in time $t'<t$ is the e^+e^- field described by the function

$$\psi(x')=\left(c_1u^{(1)}(\vec{p})+c_2u^{(2)}(\vec{p})+c_3v^{(1)}(\vec{p})+c_4v^{(2)}(\vec{p})\right)\varphi(x')$$

Using (7) in time t the field function will be:

$$\psi(x)=\int dx'G_F(x-x')\psi(x')=\left(c_1'u^{(1)}(\vec{p})+c_2'u^{(2)}(\vec{p})\right)\int dx'\varphi(x')\exp\left[-ip(x-x')\right]$$

Due to the orthogonality $\bar{u}^{(s)}v^{(t)}=0$ – the result does not contain the states with negative energy.

The condition OP2. Let us assume that $t<t'$ then $\int_K dp_0\cdots$ converges to 0 for $R\rightarrow\infty$, if the half-circumference K is enclosed in the upper half-plane and hence in this case the pole $p_0=-E$ is circumvented (see Fig. 4). In this case:

$$\int_{-\infty}^{\infty}dp_0\cdots=\lim_{\varepsilon\to0}\int_C dp_0\,\frac{(\hat{p}+m)\exp[-ip_0(t-t')]}{(p_0-E+i\varepsilon)(p_0+E-i\varepsilon)}=2\pi i\,\frac{\left(-E\gamma^0-\vec{p}\cdot\vec{\gamma}+m\right)\exp[iE(t-t')]}{-2E} \tag{18}$$

It will lead to the Green function:

$$\begin{aligned}G_F(x-x')&=\frac{i}{(2\pi)^3}\int d\vec{p}\,\frac{\left(-E\gamma^0-\vec{p}\cdot\vec{\gamma}+m\right)}{-2E}\exp[iE(t-t')+i\vec{p}(\vec{x}-\vec{x}')]\\ &\xrightarrow{\vec{p}\to-\vec{p}}\;=\frac{i}{(2\pi)^3}\int d\vec{p}\,\frac{(-\hat{p}+m)}{2E}\exp[ip(x-x')]\end{aligned} \tag{19}$$

In general for the Green function of the spinor field ($e^+ e^-$) one can write:

$$G_F(x-x') = \begin{cases} \dfrac{i}{(2\pi)^3}\displaystyle\int \dfrac{d\vec{p}}{2E}(\hat{p}+m)\exp[-ip(x-x')] & t>t' \\ \dfrac{i}{(2\pi)^3}\displaystyle\int \dfrac{d\vec{p}}{2E}(-\hat{p}+m)\exp[ip(x-x')] & t<t' \end{cases} \tag{20}$$

In the both cases for $t>t'$ as well as for $t<t'$, the poles are circumvented correctly if we make a replacement: $E \rightarrow E\text{-}i\varepsilon$. Hence

$$G_F(x-x') = \frac{1}{(2\pi)^4}\int d^4p \frac{\hat{p}+m}{p^2-m^2+i\varepsilon}e^{-ip(x-x')} \quad \Rightarrow \quad iS_F(p) = i\frac{\hat{p}+m}{p^2-m^2+i\varepsilon} \tag{21}$$

The relation (21) shows the connection between Green function a propagator. The propagator is a Fourier picture of Green function in momentum space.

Remark. When using Cauchy's theorem for calculation of the integral $\int_{-\infty}^{\infty} dp_0 \cdots$ for $t>t'$ the half-circumference K should be enclosed in the lower half-plane because only in this case $\int_K dp_0 \cdots = 0$ and due to the same reasons in the upper half-plane for $t<t'$. When we do the replacement: $p_0 = \pm E \quad \rightarrow \quad \pm(E-i\varepsilon)$,

The pole with positive energy ($+E-i\varepsilon$) (connected with electron) automatically falls into the lower half-plane and the pole with negative energy ($-E+i\varepsilon$) (connected with positron) automatically falls into the upper half-plane. Hence the replacement automatically provides that the poles are circumvented correctly.

Remark. The change of the pole from E to $E\text{-}i\varepsilon$ practically means that we replaced the stable particle with energy E by unstable particle with energy **E** and the life time $\tau=1/\varepsilon$. As the evolution factor is: $e^{-i(E-i\varepsilon)t} = e^{-iEt}\cdot e^{-\varepsilon t} = e^{-iEt}\cdot e^{-t/\tau}$.