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Amplitude of scattering in 2nd order of perturbative theory

Scattering of electron on static potential

Let us consider the scattering electron in field of static potential (e.g. atomic nucleus). The transition amplitude of electron from initial state to final state is in 1st order is reads:

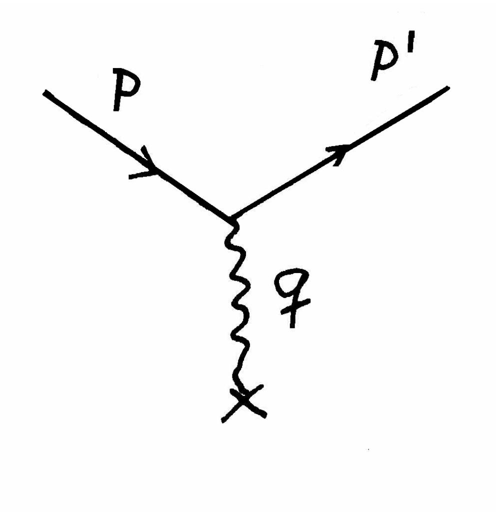


Fig. 1: Scattering of electron on static potential (1st order), electron exchanges virtual photon with scattering center.

$$T_{fi} = -i\int dx\, j_\mu^{fi}(x)\cdot A^\mu(x)$$

Using $j_\mu^{fi}(x) = -e\bar{u}_f\gamma^\mu u_i\, e^{-iqx}$ we get:

$$T_{fi} = ie\bar{u}_f\gamma^\mu u_i \int d^4x A_\mu\, e^{-iqx} = ie\bar{u}_f\gamma^\mu u_i\, A_\mu(q) \quad (1)$$

where $A_\mu(q)$ is Fourier pictures of $A_\mu(x)$. In case of the static potential $A_\mu(x)$ does not depend from time:

$$\begin{aligned} A_\mu(q) &= \int dt\, e^{-i(E_f - E_i)t}\int d^3\vec{x} A^\mu(x)\, e^{i\vec{q}\vec{x}} \\ &= 2\pi\delta(E_f - E_i)\, A_\mu(\vec{q}) \end{aligned} \quad (2)$$

For determination of 3-dimensional Fourier picture $A^\mu(q)$ we use the Maxwell equations:

$$\nabla^2 A^\mu(\vec{x}) = -j^\mu(\vec{x}) \quad (3)$$

This leads to

$$\begin{aligned} j^\mu(\vec{q}) &= \int d^3x\, j^\mu(\vec{x})e^{i\vec{q}\cdot\vec{x}} &&= -\int d^3x\left(\nabla^2 A^\mu(\vec{x})\right)e^{i\vec{q}\cdot\vec{x}} \\ \text{per partes} \Rightarrow\ &= \vec{q}^{\,2}\cdot\int d^3x\, A^\mu(\vec{x})e^{i\vec{q}\cdot\vec{x}} &&= \vec{q}^{\,2}\cdot A^\mu(\vec{q}) \end{aligned} \quad (4)$$

The result for the amplitude of electron scattering in the 1st order reads

$$T_{fi} = 2\pi\delta(E_f - E_i)\cdot e\bar{u}_f\gamma_\mu u_i \cdot \frac{1}{|\vec{q}|^2}\cdot j^\mu(\vec{q}) \quad (5)$$

The presence of δ–function leads to $q_0=0$ and hence $q^2 = -|\vec{q}|^2$. Let us assume that the static potential is the potential of atomic nucleus with the charge Ze. Then the invariant amplitude e^-Ze–scattering reads:

$$-iM = ie\bar{u}_f\,\gamma^{\mu}u_i\cdot\frac{-ig_{\mu\nu}}{q^2}\cdot\left(-i\,j^{\nu}\left(\vec{q}\right)\right) = ie\bar{u}_f\,\gamma^{0}u_i\cdot\frac{-i}{q^2}\cdot\left(-iZe\right) \tag{6}$$

where $\quad j^0(\vec{x}) = Ze\delta(x) \quad , \quad \vec{j}(\vec{x}) = \vec{0}$

The cross section of $e^{-}Ze$–scattering (Rutherford scattering) in first order:

$$\frac{d\sigma}{d\Omega} \sim |M|^2 \sim \frac{\alpha^2}{\sin^4\frac{\theta}{2}}; \quad \alpha = \frac{e^2}{4\pi} \tag{7}$$

Rutherford scattering in 2nd order of perturbative approach (approximation e4)

Into $e^{-}Ze$–scattering it is needed to include also diagrams with 4 vertexes. The first of all we will treat Rutherford scattering in case of fluctuation of virtual photon to pair $e^{+}e^{-}$ (see Fig. 2).

We expect that after inclusion of the 2nd order of perturbative method the amplitude of process is:

$$M = \alpha A^{(1)} + \alpha^2 A^{(2)}, \quad \alpha = \frac{e^2}{4\pi} \quad ,$$

as $\alpha \cong 1/137$, it is expected that the 2nd order will be suppressed by around 100-times. It is true provided that the spinor part of amplitudes ($A^{(1)}, A^{(2)}$) do not differs a lot. We will show that direct calculation of $A^{(2)}$ leads to divergence of this quantity.

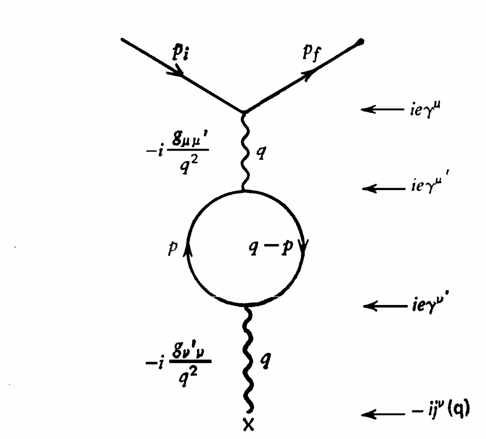


Fig. 2:Rutherford scattering in 2nd order – fluctuation of virtual photon to pair $e^{+}e^{-}$.

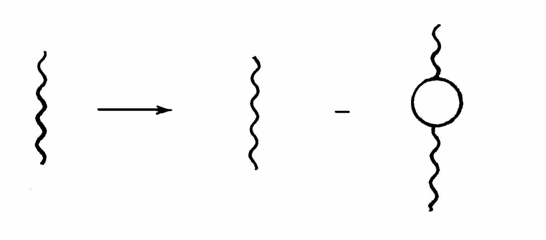
Invariant amplitude:

$$-iM^{(2)} = -ie\bar{u}_f\,\gamma^{\mu}u_i \cdot\frac{-ig_{\mu\mu'}}{q^2}$$

$$\cdot\int d^4p\left[\left(ie\gamma^{\mu'}\right)_{\alpha\beta}\cdot\frac{(\hat{p}+m)_{\beta\delta}}{p^2-m^2}\cdot\left(ie\gamma^{\nu'}\right)_{\delta\lambda}\cdot\frac{(\hat{q}-\hat{p}+m)_{\lambda\alpha}}{(q-p)^2-m^2}\right]\cdot\left(-i\frac{g_{\nu'\nu}}{q^2}\right)\left(-i\,j^{\nu}\left(\vec{q}\right)\right) \tag{8}$$

The addition of amplitude $M^{(2)}$ $(\sim e^4)$ to amplitude $M^{(1)}$ $(\sim e^2)$ is equivalent to the folowing modification of propagator (see Fig. 3):

Fig. 3: Modification of propagator that enable to include the effect of virtual pairs into amplitude of scattering.



$$\frac{-ig_{\mu\nu}}{q^2} \quad \to \quad \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\mu'}}{q^2} \cdot I^{\mu'\nu'}(q^2) \cdot \frac{-ig_{\nu'\nu}}{q^2} \quad = \quad \frac{-ig_{\mu\nu}}{q^2} + \frac{-i}{q^2} \cdot I_{\mu\nu}(q^2) \cdot \frac{-i}{q^2} \tag{9}$$

where

$$I_{\mu\nu}(q^2) = \int \frac{d^4 p}{(2\pi)^4} Tr\left[(ie\gamma_\mu) \cdot \frac{(\hat{p}+m)}{p^2 - m^2} \cdot (ie\gamma_\nu) \cdot \frac{(\hat{q}-\hat{p}+m)}{(q-p)^2 - m^2} \right] \tag{10}$$

The problem is that the integral $I_{\mu\nu}(q^2)$ diverge at $p \to \infty$ (at calculation of $I_{\mu\nu}$ is needed to calculate traces of γ-matrices (see Appendix A) and then calculate the integral (see appendix B).

The calculation of the integral $I_{\mu\nu}(q^2)$ gives:

$$I_{\mu\nu}\left(q^2\right) = g_{\mu\nu} I\left(q^2\right) + \underbrace{\cdots}_{\text{terms} \sim q_\mu q_\nu} \tag{11}$$

where

$$I\left(q^2\right) = \underbrace{\frac{\alpha}{3\pi} \int_{m^2}^{\infty} \frac{dp^2}{p^2}}_{\text{logaritmically diverges}} \underbrace{- \frac{2\alpha}{\pi} \int_0^1 dz \cdot z(1-z) \ln\left(1 - \frac{q^2 z(1-z)}{m^2}\right)}_{\text{finite part}} \; , \tag{12}$$

where m is the electron mass. The terms proportional to $q_\mu q_\nu$ give after summation the zero contribution to the process amplitude. It is a consequence of calibration invariance of the electromagnetic interaction that leads to the conservation law of charge: $\partial_\mu J^\mu(x) = 0$ $\to$ $q_\mu j^\mu(p) = 0$ (where q_μ is 4-momentum of photon that is coupled to the current j^μ) and amplitude $M^{(2)}$ has the structure: $M^{(2)} \sim j^\mu I_{\mu\nu} j^\nu$. The problem we will try to solve in such a way that in the first we will regularize logarithmically divergent part of $I(q^2)$, i.e. in $I(q^2)$ we will change: $\infty \quad \rightarrow \quad M^2$

Remark: *Regularization means the upper limitation of the momentum „circulating" in loop. This limitation means under the Heisenberg principle of uncertainty a discrete structure of space at a level* $1/M$. *If as the limit* M *we took Planck mass* ($M_P = \sqrt{\hbar c/G} = 1.22\cdot 10^{19} GeV/c^2$) *then discreetness of space is represented by the minimal length* $l_p = \hbar/(M_P c) \approx 1.6\cdot 10^{-35} m$.

The effects of e^+e^-–loops at small transferred momenta

At small momenta $(-q^2 \ll m^2)$ it is valid:

$$\ln\left(1-\frac{q^2 z(1-z)}{m^2}\right) \quad \approx \quad -\frac{q^2 z(1-z)}{m^2} \tag{13}$$

In this case the $I(q^2)$ reads:

$$I(q^2) \quad \approx \quad \frac{\alpha}{3\pi}\ln\frac{M^2}{m^2} + \frac{\alpha}{15\pi}\frac{q^2}{m^2} \tag{14}$$

The effects of e^+e^-–loops at large transferred momenta

At large momenta $(-q^2 \gg m^2)$ it is valid:

$$\ln\left(1-\frac{q^2 z(1-z)}{m^2}\right) \quad \approx \quad \ln\left(\frac{-q^2}{m^2}\right) \tag{15}$$

and

$$I(q^2) \quad \approx \quad \frac{\alpha}{3\pi}\ln\left(\frac{M^2}{m^2}\right) - \frac{\alpha}{3\pi}\ln\left(\frac{-q^2}{m^2}\right) \quad = \quad \frac{\alpha}{3\pi}\ln\left(\frac{M^2}{-q^2}\right) \tag{16}$$

The amplitude of Rutherford scattering at small transverse momenta

If into the relation (11) we put instead of $I(q^2)$ the expression (14) and in the amplitude (6) we replace the propagator under the scheme (9) then the amplitude in the 2nd order reads:

$$-iM = ie\bar{u}_f \gamma^0 u_i \cdot \left(\frac{-i}{q^2}\right)\cdot\left(1 - \frac{e^2}{12\pi^2}\ln\left(\frac{M^2}{m^2}\right) - \frac{e^2}{60\pi^2}\frac{q^2}{m^2} + O(e^4)\right)(-iZe) \tag{17}$$

Let us introduce the reduced charge in the following way:

$$e_R = e\left(1-\frac{e^2}{12\pi^2}\ln\left(\frac{M^2}{m^2}\right)\right)^{\frac{1}{2}} \tag{18}$$

In this case with a precision $O(e^4)$ for $(-iM)$ we get:

$$-iM = ie_R\bar{u}_f\gamma^0 u_i\cdot\left(\frac{-i}{q^2}\right)\cdot\left(1-\frac{e_R^2}{60\pi^2}\frac{q^2}{m^2}\right)\left(-iZe_R\right) \tag{19}$$

Let us assume that the charge measured in experiment at $q^2\approx 0$ (Thompson scattering) is the reduced charge e_R then the amplitude M is finite in the 2nd order.

Remark. The charge e that we have used to express the amplitude M, is the charge that is "sitting" in the equation of motion (for motion of electron in electromagnetic field). At the same time the equation of motion we can treat as the Lagrange equation of a certain physical system of fields (in our case this system contains the electron-positron field and the electromagnetic one) with a certain Lagrangian – in our case: the Lagrangian of QED:

$$L_{QED} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + eQA_\mu\cdot\bar{\psi}\gamma^\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}).$$

The charge e „sitting" in Lagrangian we will call the bare charge that differs from the experimentally determined charge.

Lamb shift

The first term in the amplitude of e^-Ze-scattering (19) is connected with the Coulomb potential. We can be convinced in this by backward Fourier transformation:

$$V_0(r) \quad=\quad Ze_R^2\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{\exp(i\vec{q}\cdot\vec{r})}{|\vec{q}|^2} \quad=\quad \frac{Ze_R^2}{4\pi r} \tag{20}$$

$$\left(\quad d^3\vec{q} = 2\pi|\vec{q}|^2 d|\vec{q}|\sin\vartheta d\vartheta \quad,\quad \vec{q}\cdot\vec{r} = |\vec{q}|r\cdot\cos\vartheta \quad\right)$$

The 2nd term in the amplitude of e^-Ze-scattering (19) corresponds to the effect of virtual e^+e^--pair in propagator:

$$\frac{Ze_R^4}{60\pi^2 m^2}\cdot\int\frac{d^3\vec{q}}{(2\pi)^3}\exp(i\vec{q}\cdot\vec{r}) \quad=\quad \frac{Ze_R^4}{60\pi^2 m^2}\delta(\vec{r}) \tag{21}$$

The potential acting between the charge of nucleus and electron is:

$$V(r) = \underbrace{-\frac{Ze_R^2}{4\pi r}}_{\text{Coulomb contribution}} - \underbrace{\frac{Ze_R^4}{60\pi^2 m^2}\delta(\vec{r})}_{\text{contribution of virtual pair } e^+e^-} \tag{22}$$

Remark: Eq. 22 is not exact as we used approximate value of $I(q^2)$ at small $(-q^2)$.

It should be noted that the contribution of virtual pair in photon propagator manifests itself experimentally – it leads to the shift of the energetic levels in hydrogen atom i.e. to the so-called Lamb shift. Using perturbative approach the additional term leads to the following shift of energetic levels:

$$\Delta E_{nl} = -\frac{e_R^4}{60\pi^2 m^2} \cdot \left|\Psi_{nl}(0)\right|^2 \cdot \delta_{l0} \tag{23}$$

Where $\Psi_{nl}(0)$ is the value of the wave function describing the state of hydrogen atom with the main quantum number n and orbital number l in the centre of atom i.e. in nucleus.

The Lamb shift is really observed in the spectra of hydrogen atom!

Anomalous magnetic moment of electron

The full set of diagrams ~ e^4 (2nd order of perturbative theory) for the e^-Ze-scattering is in Fig. 4.

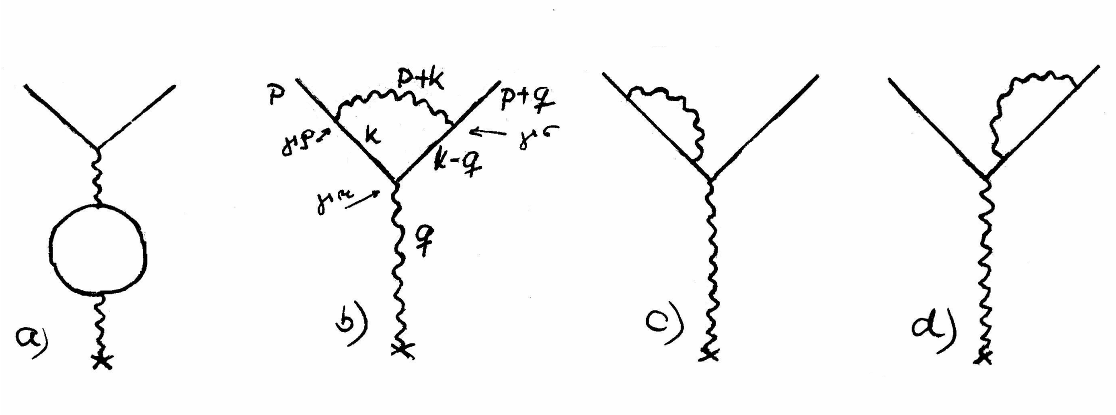


Fig. 4: Diagrams for the ***eZe***-scattering in 2nd order of perturbative theory.

The loops in diagrams *b*, *c* and *d*, similarly as in diagram *a*, also diverge. These divergences could be „hidden“ in redefined charge, mass and wave function of electron.

Let us consider the diagram *b*, the loop in vertex modifies the structure of electron current ($-e\bar{u}_f\gamma^\mu u_i$):

$$-e\bar{u}_f\gamma^\mu u_i \to -e\bar{u}_f\left(\gamma^\mu+\Lambda^\mu\right)u_i =$$

$$= i\bar{u}_f\left(ie\gamma^\mu\right)u_i + i\bar{u}_f\left((ie)^3\int\frac{d^4k}{(2\pi)^4}\cdot\frac{-ig_{\rho\sigma}}{(k+p)^2}\gamma^\rho\frac{i\left(\hat{k}-\hat{q}+m\right)}{(k-q)^2-m^2}\gamma^\mu\frac{i\left(\hat{k}+m\right)}{k^2-m^2}\gamma^\sigma\right)u_i \tag{24}$$

In the e^4 approximation and for small $(-q^2)$ the loop integral reads:

$$I_{(4b)} \approx -e\bar{u}_f\left\{\underbrace{\gamma^\mu\left[1+\frac{\alpha}{3\pi}\frac{q^2}{m^2}\left(\ln\frac{m}{m_\gamma}-\frac{3}{8}\right)\right]}_{\text{modifikuje náboj}}-\underbrace{\left[\frac{\alpha}{2\pi}\frac{i\sigma_{\mu\nu}}{2m}q^\nu\right]}_{\text{modifikuje el.tok}}\right\}u_i \tag{25}$$

Remark. The m_γ is small effective mass of photon that was introduced for removing divergences at small **|k|** (circulating momentum) – under the Heisenberg principle to m_γ correspond a length 1/ m_γ that is an upper limit for the wave length of photon - it could means a finite dimension of universe. Ward identity (see below) will show us that the contribution of diagram (*b*) modifying the charge will be canceled by the contributions of the diagrams (c) and (d), so we can ignore it in (25).

Let us make the Gordon expansion of electromagnetic current:

$$-e\bar{u}_f\gamma^\mu u_i = -e\bar{u}_f\left(\frac{\left(p_f+p_i\right)^\mu}{2m}-i\frac{\sigma^{\mu\nu}q_\nu}{2m}\right)u_i \tag{26}$$

The term $\sigma^{\mu\nu}q_\nu$ describes magnetic moment of electron:

$$\vec{\mu} = -\frac{e}{2m}\vec{\sigma} = -g\frac{e}{2m}\vec{S} \tag{27}$$

where $\vec{S}=\vec{\sigma}/2$ a ***g=2*** is gyromagnetic factor.

By inserting the Gordon expansion into the amplitude (25), the magnetic moment reads:

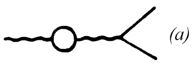
$$\vec{\mu} = -\frac{e}{2m}\left(1+\underbrace{\frac{\alpha}{2\pi}}_{\text{anomalous mag. moment}}\right)\vec{\sigma} \tag{28}$$

Hence $g=2+\alpha/\pi$. If we include also the higher orders of perturbative expansion, we will achieve an excellent agreement with experiment:

$$\left(\frac{g-2}{g}\right)_{theory} = (1159655.4 \pm 3.3)\cdot 10^{-9}$$
$$\left(\frac{g-2}{g}\right)_{exper} = (1159657.7 \pm 3.5)\cdot 10^{-9} \qquad (29)$$

Ward identity

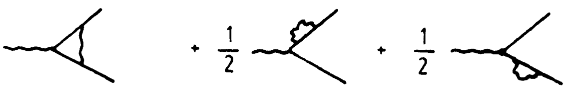
A complete calculation of the amplitude requires include into $\boldsymbol{e_R}$ (in the e^4 approximation) also the infinite parts of the loops shown in Fig. 4.

Its clear that the diagram 

Does not depend on type of scattering particle.

However in case of the diagrams (*b*), (*c*) and (*d*) the scattering particle is a part of loops. As these loops give contributions to the charge correction there exists a threat that the charge will depend on type of particle – e.g. the charge ($\boldsymbol{e^-}$) ≠ náboj ($\boldsymbol{\mu^-}$). However the **dependence of charge on type of particle is not observed experimentally!**

By the direct calculation of diagrams (*b*), (*c*) and (*d*) it can be shown that the Ward identity is valid:



$\Big|_R = 0$ (30)

Hence modification of charge is carried out only due to the polarization of vacuum (diagram *a*).

Scattering of the $e^-\mu^- \to e^-\mu^-$ in e^4 approximation

The amplitude of the $e^-\mu^-$ in e^4 approximation we can get from the amplitude of e^-Ze scattering by the replacement: $-ij^\mu(q) \equiv (Ze,0) \quad \to \quad ie\bar{u}_f(p')\gamma^\mu u_i(p)$ (see Fig. 5)

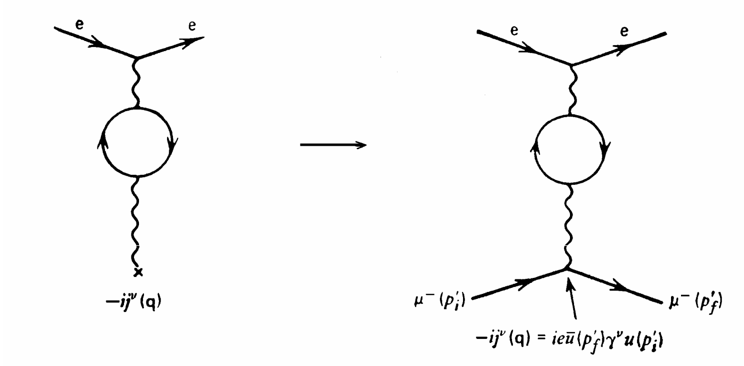


Fig. 5: Transition from the e^-Ze-scattering to the $e\mu$-scattering.

If we include into the $e\mu$-scattering the e^4-diagrams then we need to work with renormalization charge e_R. The correction to propagator is the same as in case of the e^-Ze-scattering:

$$\frac{(-i)}{q^2} \cdot I_{\mu\nu} \cdot \frac{(-i)}{q^2} \;,$$

And it will be the same in all process with photon as an intermediate particle. At the $e\mu$-scattering however exist also other e^4-diagrams that should be taken into account.

Renormalization

Charge is connected with photon-electron interaction – characterizes its power (Fig. 6):

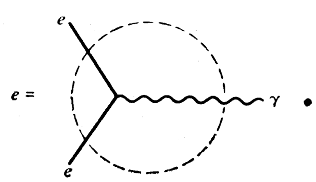


Fig. 6: Photon - electron coupling is characterized by charge ***e***.

However the $e\gamma$-interaction goes also through diagrams as those shown in Fig. 7.

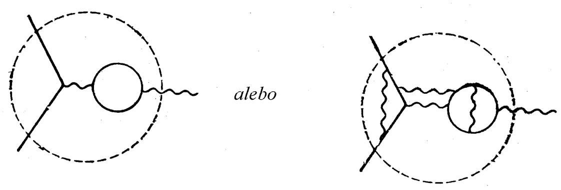


Fig. 7: Examples of the $e\gamma$-interaction characterized by more complex diagrams.

In Coulomb experiment the charge is measured as contribution of all diagrams. The charge in the diagram in *Fig. 6* we will denote as the "bare" charge. This charge is not identical with that measured in experiment at a given transferred momentum. The relation of the charge measured in experiment (***e***) and bare charge ($\boldsymbol{e_0}$) is given by the diagram expression shown in *Fig.8*:

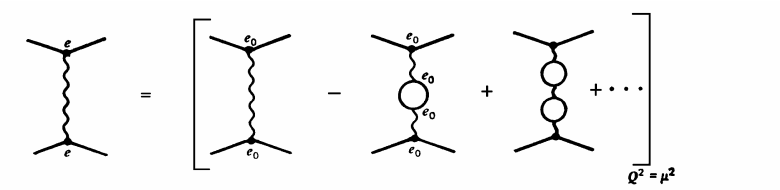


Fig.8: relation between the charge ***e*** measured in experiment at $Q^2=\mu^2$ and bare charge $\boldsymbol{e_0}$.

Expression of the experimental charge through the bare one assumes that experiment is carried out at the transferred momentum $q^2 = -Q^2 = -\mu^2$.

Remark. It should be taken into account that at measurement of charge in Coulomb experiment, experimenter assumes that the process is described by one-photon amplitude A_0 (diagram on left in Fig. 8) in that „seeds" the experimental charge – the cross section of process is proportional to 4th power of the experimental charge:

$$\sigma_{coul} = e^4 k_{flux} \left|A_0\right|^2 k_{ph.space} = e^4 \times F_{kin},$$

As the factor of input current (k_{flux}) and that of phase space ($k_{ph.space}$) are known quantities by measurement of cross section we measure charge. The measured charge represents an effective charge because at its extraction from cross section, experimenter does not take into account the higher orders.

If in the relation between ***e*** and $\boldsymbol{e_0}$ we restrict ourselves to one-loop approximation:

$$e^2 = e_0^2\left[1 - I(q^2 = -\mu^2) + O(e_0^4)\right] \tag{31}$$

or after having made the square (we take the $I(q^2)$ as a small quantity)

$$e = e_0\left[1 - \tfrac{1}{2} I(q^2 = -\mu^2) + O(e_0^4)\right] \quad (32)$$

or expressed by diagrams:

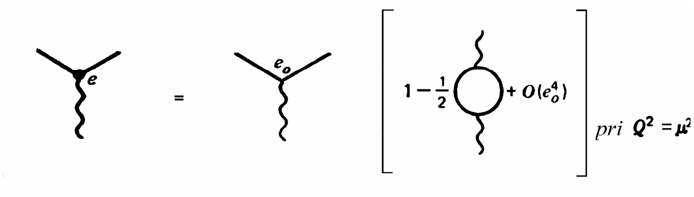


Fig. 9: The relation between experimental and bare charge in 1-loop approximation.
The inverse relation in one-loop approximation:

$$e_0 = e\left[1 + \tfrac{1}{2} I(q^2 = -\mu^2) + O(e_0^4)\right] \quad (33)$$

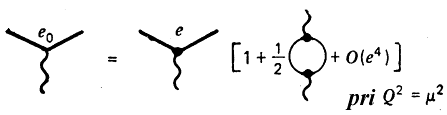


Fig. 10: Bare charge vs. experimental one in 1-loop approximation

We have found the relation between ***e*** (experimentally determined charge at transferred momentum μ) and e_0 (bare charge). Let us express now the amplitude of ***eμ***-scattering ($M(e_0)$) for an arbitrary square of transferred momentum Q^2 using the bare charge e_0.

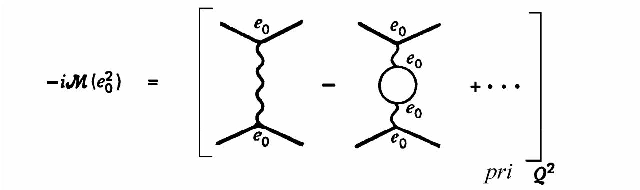


Fig. 11: Amplitude of scattering expressed through bare charge.

Expressing the e_0 through the ***e*** we get the amplitude of ***eμ***-scattering ($M(e^2)$) expressed through the experimental charge at fixed $Q^2 = \mu^2$:

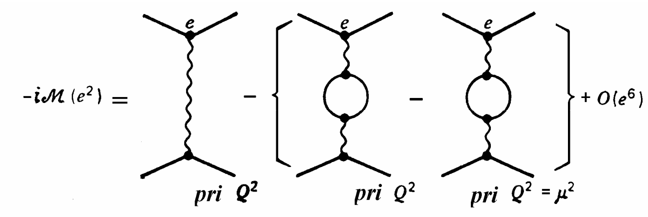
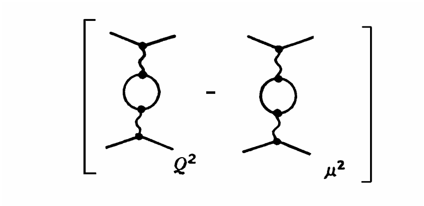


Fig. 12: Amplitude of eμ-scattering expressed through experimental charge at transferred momentum μ^2 in 2nd order of perturbative expansion.

Owing due to the fact that at transition from $M(e_0^2)$ to $M(e^2)$ we in frame of 1-loop approximation nothing have thrown away, it should be fulfilled :

$$M(e^2) = M(e_0^2). \quad (34)$$

At the same time the term of order e^4 consists from 2 parts – one them contains a loop at Q^2 and the other at μ^2 and (in approximation of big transferred momenta) is valid:



$$\sim \frac{\alpha}{3\pi}\ln\frac{M^2}{Q^2} - \frac{\alpha}{3\pi}\ln\frac{M^2}{\mu^2} = \frac{\alpha}{3\pi}\ln\frac{\mu^2}{Q^2} \qquad (35)$$

Hence the difference of both parts is finished! And we see that the expansion of the amplitude of $e\mu$-scattering into series in powers of experimental charge e gives the finite coefficients at powers of e (unlike of the expansion is done in powers of bare charge).

The free parameter μ that have the dimension of mass (transferred momentum at which the charge is defined) arises as a consequence of renormalization of charge.

A different choice of scale μ leads to different expansions of amplitude M. However $|M|^2$ is a quantity proportional to cross section therefore cannot be dependent on choice of the scale μ. This independence is expressed by the equation of renorm-group:

$$\mu\frac{dM}{d\mu} = \left(\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial e}{\partial\mu}\frac{\partial}{\partial e}\right)M = 0 \qquad (36)$$

Hence the explicit dependence of M from μ is compensated by a dependence of e from μ^2.

Running coupling constant - screening of charge in QED

The effect of modification of charge it is possible to determine in all orders of perturpative expansion what can be expressed by means of the diagrams:

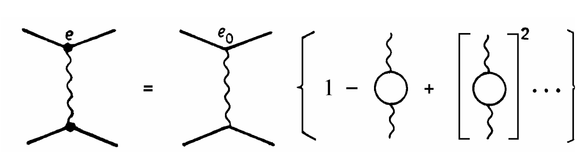
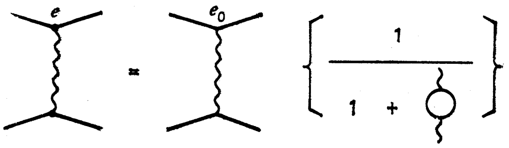


Fig. 13: Charge e at a given Q^2 expressed through bare charge e^0 by inclusion of all orders.

The above mentioned geometrical series can be summed as follows:



$$\equiv \quad e^2(Q^2) = e_0^2\left[\frac{1}{1+I(-Q^2)}\right] \tag{37}$$

This relation means that experimentally determined charge depends on value of square of transferred momentum Q^2 at which this charge is determined. The quantity $\alpha(Q^2) = \frac{e^2(Q^2)}{4\pi}$ is called the running coupling constant and in limit at big Q^2 it reads:

$$I(-Q^2) \approx -\frac{\alpha_0}{3\pi}\ln\frac{Q^2}{M^2} \tag{38}$$

From where we get:

$$\alpha(Q^2) = \frac{\alpha_0}{1-\frac{\alpha_0}{3\pi}\ln\frac{Q^2}{M^2}} \tag{39}$$

If we introduced renormalization scale μ (i.e. reference transferred momentum at which the charge was determined) and express α_0 through $\alpha(\mu^2)$ we get:

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1-\frac{\alpha(\mu^2)}{3\pi}\ln\frac{Q^2}{\mu^2}} \tag{40}$$

At the expression $\alpha(Q^2)$ we get rid of the parameter ***M*,** however we have introduced the scale μ.

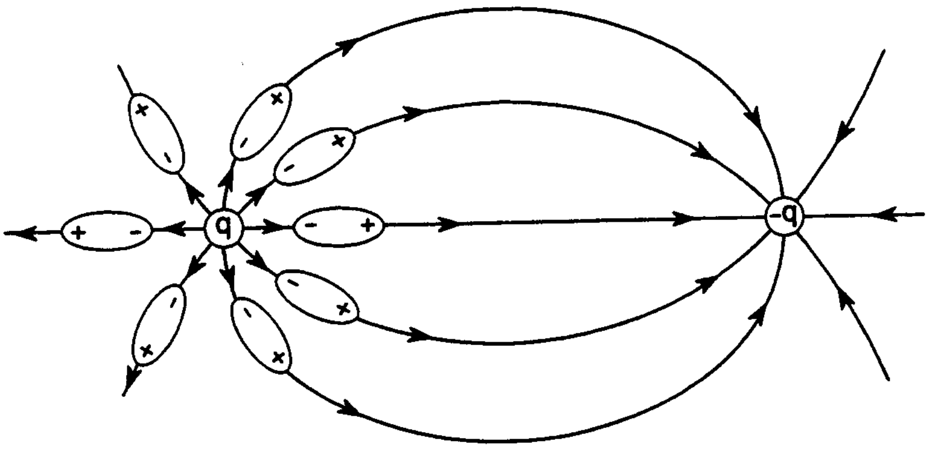


Fig. 14: Effect of e^-e^+ loops at interaction of two charges.

The running coupling constant $\alpha(Q^2)$ describes dependence of effective charge on distance ($1/Q$) between charged particles. With increasing Q^2 photon „see“ bigger and bigger charge. From Eq.39 it is clear that the bare charge is the charge at the „limit“ transferred momentum M (or at the minimal possible distance between charges $1/M$). If we go with $M \to \infty$ then the bare charge becomes infinite (as is seen from (39)).

Remark. We took into account only the e^-e^+ –loops, however at large Q^2 their contribution will give also other loops – the loops from pairs $\mu^-\mu^+$, $q\bar{q}$, etc.

Coupling constant of strong interaction

Before discussion about the coupling constant of strong interaction it is needed to discuss the question of real and virtual photons.

Real and virtual photons

The amplitude of $e\mu$-scattering (in general in scattering of particles *A* and *B*) reads:

$$\begin{aligned} T_{fi} &= -i\int j_\mu^A(x)\cdot\left(\frac{-g_{\mu\nu}}{q^2}\right)\cdot j_\nu^B(x)\cdot d^4x \\ &= -i\int\left(\underbrace{\frac{j_1^A j_1^B + j_2^A j_2^B}{q^2}}_{\text{priečny fotón}} + \underbrace{\frac{j_3^A j_3^B - j_0^A j_0^B}{q^2}}_{\text{skalárny fotón}}\right)d^4x \end{aligned} \qquad (41)$$

If we choose the coordinate system in such a way that $\vec{q}$ (transferred momentum) is in direction of z-axis, i.e. $q^\mu = \left(q^0, 0, 0, |\vec{q}|\right)$, then for the continuity equation we have:

$$q^\mu j_\mu = q^0 j_0 - |\vec{q}|\, j_3 = 0 \qquad (42)$$

If the exchanged photon is almost real ($q^0 \approx |\vec{q}|$) then $j_0 \approx j_3$ and the contributions of longitudinal and scalar photons are mutually cancelled. For the truly virtual photon:

$$\begin{aligned} j_3 &= \frac{q_0}{|\vec{q}|}\, j_0 \\ T_{fi} &= -i\int\left(\underbrace{\frac{j_1^A j_1^B + j_2^A j_2^B}{q^2}}_{T} + \underbrace{\frac{j_0^A j_0^B}{|\vec{q}|^2}}_{C}\right)\cdot d^4x \end{aligned} \qquad (43)$$

T: propagation of virtual photons in states with transverse polarization (time-like propagator).

C: describes immediate Coulomb interaction between the charges of particles *A* and *B*.

Using the inverse Fourier transformation for the part *C* we get:

$$\frac{1}{|\vec{q}|^2} = \int d^3\vec{x} \cdot \exp(i\vec{q}\cdot\vec{x}) \cdot \frac{1}{4\pi|\vec{x}|}, \tag{44}$$

We see that *C* does not depend on time and it leads to:

$$T_{fi}^{coul} = -i\int dt \int d^3\vec{x}_1 \int d^3\vec{x}_2 \frac{j_0^A(t,\vec{x}_1)\, j_0^B(t,\vec{x}_2)}{4\pi|\vec{x}_2 - \vec{x}_1|} \tag{45}$$

Hence the charges of particles *A* and *B* interact without a delay in time ***t***.

Running coupling constant for QCD

A basic principle of determination of the strong coupling constant $\alpha_S(Q^2)$ as a function of Q^2 is the same as in QED however the result is diametrically different. In 2nd order (for QCD) we have:

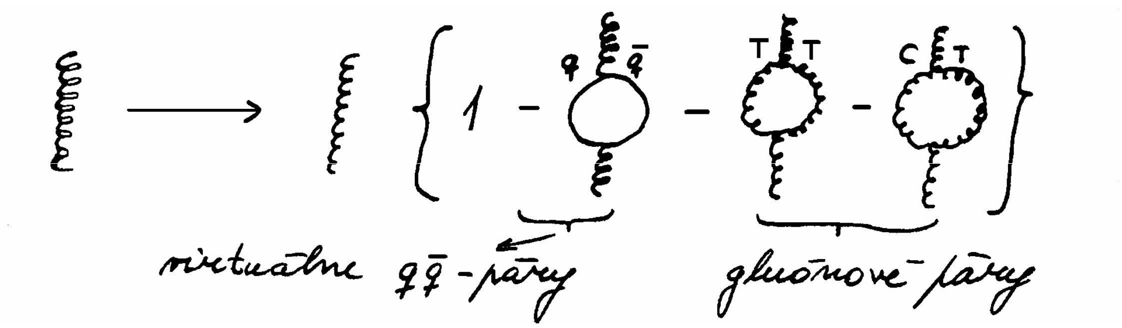


Fig. 15: Effect of virtual pairs (gluons and quarks) in QCD, T is transversal gluon and C is „Coulomb“ gluon.

The shown diagrams give at $\ln Q^2/\mu^2$ in equation for the coupling constant (see Eq. 40) the coefficient:

$$\frac{\alpha_S(\mu^2)}{4\pi}\left(\underbrace{-\frac{2}{3}n_f}_{q\bar{q}} \underbrace{-5}_{TT} \underbrace{+16}_{CT}\right), \tag{46}$$

where n_f is the number of the quark flavors.

Remark: In QED $n_f = 1$ (only *1* pair of $e^+ e^-$) the corresponding coefficient is $-4/3$, as the definitions of α and α_s differs by a factor 2 (given historically).

In general it is possible to prove the theorem:

Arbitrary states that can arise at time-like propagator lead to screening of charge, hence to the negative coefficient at $\ln\left(Q^2/\mu^2\right)$. The Coulomb gluons (space-like propagator) lead to the opposite phenomenon.

The coupling constant for large Q^2 can be expressed (in acco0rdance with Eq. 40) as follows:

$$\alpha_S(Q^2) = \frac{\alpha_S(\mu^2)}{1 + \frac{\alpha_S(\mu^2)}{4\pi}\left(11 - \frac{2}{3}n_f\right)\ln\frac{Q^2}{\mu^2}} \tag{47}$$

$\alpha_S\left(Q^2\right)$ decreases with increasing Q^2 – is getting small at small distances:

$Q^2 \to \infty \quad \Rightarrow \quad \alpha_S\left(Q^2\right) \to 0$ (asymptotic freedom).

At small Q^2 the coupling α_S gets big – let us denote $Q^2 = \Lambda^2$ - for

$$1 + \frac{\alpha_S(\mu^2)}{4\pi}\left(11 - \frac{2}{3}n_f\right)\ln\frac{\Lambda^2}{\mu^2} = 0 \quad \Rightarrow \quad \alpha_S(\mu^2) = -\frac{4\pi}{\left(11 - \frac{2}{3}n_f\right)\ln\frac{\Lambda^2}{\mu^2}} \tag{48}$$

If we put $\alpha_S\left(\mu^2\right)$ into the expression for $\alpha_S\left(Q^2\right)$ (47) we get:

$$\alpha_S(Q^2) = \frac{4\pi}{\left(11 - \frac{2}{3}n_f\right)\ln\frac{Q^2}{\Lambda^2}} \tag{49}$$

From where one can see that at Q^2 $\left(Q^2 \gg \Lambda^2\right)$ $\alpha_S\left(Q^2\right)$ is small ($\alpha_S\left(Q^2\right) \to 0$) and perturbative theory is applicable while for small Q^2 $\left(Q^2 \approx \Lambda^2\right)$ coupling constant $\alpha_S\left(\mu^2\right)$ is big and the perturbative approach cannot be applied. For big $\alpha_S\left(\mu^2\right)$ the coupling force is so big that the strength of binding is so big that quarks and gluons are coupled into clusters and form hadrons (confinement of quarks). The quantity Λ is usually denoted as Λ_{QCD} and represents a free parameter of the theory. The experimental measurements of the strong interaction coupling constant are shown in Fig. 16 and 17. In the first of them is shown the dependence of α_S on transferred momentum μ, while in the other (Fig.17) are shown the measurement of α_S at mass of Z-boson (μ=M_Z).

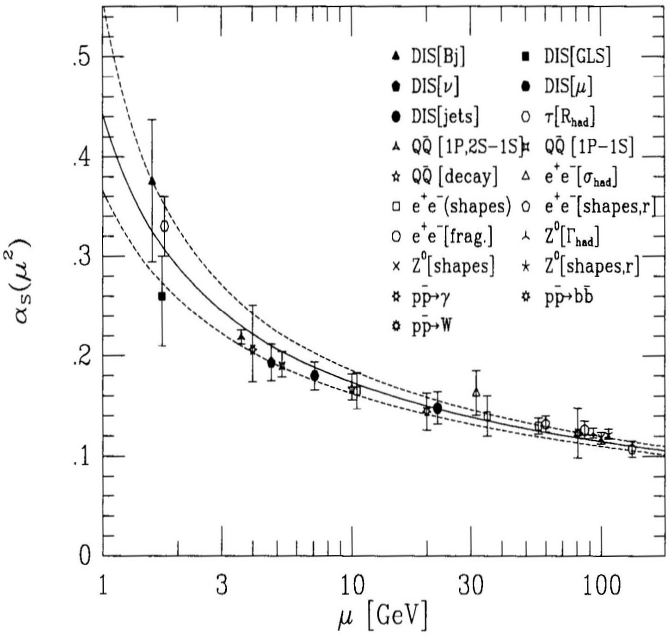


Fig. 16: The coupling constant of strong interaction as function of transferred momentum.

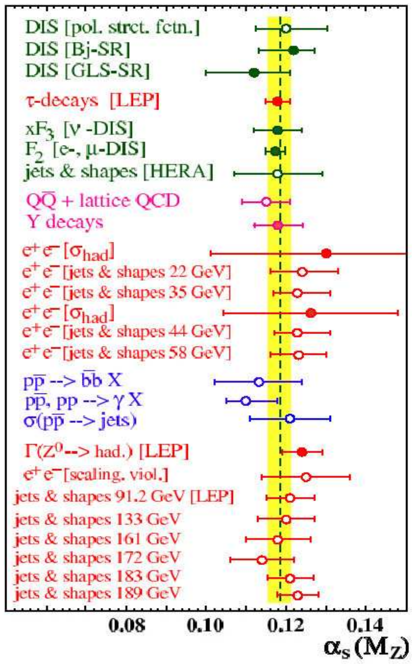


Fig. 17: The coupling constant of strong interaction measured at M_Z.

From the measurement of the coupling constant of strong interaction for Λ_{QCD} can be retrieved:

$$\Lambda_{QCD}^{(4)} = 234 \pm 26 \pm 50 \ MeV$$

$$\Lambda_{QCD}^{(5)} = 209 \begin{cases} +39 \\ -33 \end{cases} MeV$$

Index ***(n)*** in $\Lambda^{(n)}$ means the effective number of quark flavors participating in the studied process.

Appendix A

At calculation of the quantity $\boldsymbol{I_{\mu\nu}(q^2)}$ it is needed to use the following properties of traces of γ-matrices product:

$$
\begin{aligned}
Tr\left(\gamma^{\mu}\gamma^{\nu}\right) &= 4g^{\mu\nu} \\
Tr\left(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right) &= 4\left[g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}-g^{\mu\nu}g^{\rho\sigma}\right] \\
Tr\left(\gamma_5\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right) &= 4i\varepsilon^{\mu\nu\rho\sigma} \\
Tr\left(\gamma_5\gamma^{\mu}\gamma^{\nu}\right) &= 0
\end{aligned}
\qquad \text{(A.1)}
$$

And commutation relations ($\hat{a} \equiv a_{\mu}\gamma^{\mu}$) :

$$
\begin{aligned}
\hat{a}\hat{b} &= 2(ab)-\hat{b}\hat{a} \\
\hat{a}\gamma^{\mu} &= 2a^{\mu}-\gamma^{\mu}\hat{a} \\
\hat{a}\gamma^{5} &= -\gamma^{5}\hat{a}
\end{aligned}
\qquad \text{(A.2)}
$$

Appendix B: On calculation of divergent diagrams

At calculation of divergent diagrams we come from the calculation of the integral $\boldsymbol{J_2}$:

$$
J_2(p,\alpha)=\int\frac{dk}{\left(k^2-2pk+\alpha\right)^2}=i\int_0^M dk\cdot k^3\int_0^{\pi}d\chi\int_0^{\pi}d\vartheta\int_0^{2\pi}d\phi\,\frac{\sin\vartheta\cdot\sin^2\chi}{\left(k^2-2pk+\alpha\right)^2}=i\pi^2\left(\ln\frac{M^2}{\alpha-p^2}-1\right)_{M\to\infty} \qquad \text{(B1)}
$$

By integration of $J_2(p,\alpha)$ through α we get:

$$
J(p,\alpha)=\int\frac{dk}{k^2-2pk+\alpha}=i\pi^2\left[M^2-\frac{1}{2}p^2+(p^2-\alpha)\ln\frac{M^2}{\alpha-p^2}\right]_{M\to\infty} \qquad \text{(B2)}
$$

By differentiation of $J(p,\alpha)$ by α and p_{μ} one gets:

$$\int \frac{dk \cdot k_\mu}{\left(k^2 - 2pk + \alpha\right)^2} = \frac{1}{2}\frac{\partial J(p,\alpha)}{\partial p_\mu} = i\pi^2 p_\mu \left(\ln\frac{M^2}{\alpha - p^2} - \frac{3}{2}\right) \tag{B3}$$

$$\int \frac{dk}{\left(k^2 - 2pk + \alpha\right)^3} = \frac{1}{2}\frac{\partial^2 J(p,\alpha)}{\partial^2 \alpha^2} = i\pi^2 \frac{1}{\alpha - p^2} \tag{B4}$$

$$\int \frac{\boldsymbol{dk \cdot k_\mu}}{\left(\boldsymbol{k}^2 - 2\boldsymbol{pk} + \boldsymbol{\alpha}\right)^3} = -\frac{1}{4}\frac{\boldsymbol{\partial^2 J(p,\alpha)}}{\boldsymbol{\partial p_\mu \partial \alpha}} = \boldsymbol{i}\frac{\boldsymbol{\pi}^2}{2}\frac{\boldsymbol{p_\mu}}{\boldsymbol{\alpha - p}^2} \tag{B5}$$

$$\int \frac{dk}{\left(k^2 - 2pk + \alpha\right)^4} = -\frac{1}{6}\frac{\partial^3 J(p,\alpha)}{\partial \alpha^3} = i\frac{\pi^2}{6}\frac{1}{\left(\alpha - p^2\right)^2} \tag{B6}$$

$$\int \frac{dk \cdot k_\mu}{\left(k^2 - 2pk + \alpha\right)^4} = \frac{1}{12}\frac{\partial^3 J(p,\alpha)}{\partial p_\mu \partial \alpha^2} = i\frac{\pi^2}{6}\frac{p_\mu}{\left(\alpha - p^2\right)^2} \tag{B7}$$

$$\int \frac{dk \cdot k_\mu k_\upsilon}{\left(k^2 - 2pk + \alpha\right)^2} = i\pi^2 \left\{\frac{g_{\mu\upsilon}}{2}\left(\frac{M^2}{2} - \frac{5}{6}p^2 + \frac{\alpha}{2} + (p^2 - \alpha)\ln\frac{M^2}{\alpha - p^2}\right) + \left(\ln\frac{M^2}{\alpha - p^2} - \frac{11}{6}\right)p_\mu p_\upsilon\right\} \tag{B8}$$

$$\int \frac{dk \cdot k_\mu k_\nu}{\left(k^2 - 2pk + \alpha\right)^3} = \frac{1}{12}\frac{\partial^2 J(p,\alpha)}{\partial p_\mu \partial p_\nu} = i\frac{\pi^2}{4}\left[g_{\mu\nu}\left(\ln\frac{M^2}{\alpha - p^2}\right) + \frac{2p_\mu p_\nu}{\alpha - p^2}\right] \tag{B9}$$