On divergences in QED

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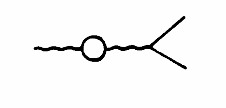
Summary of e^-Ze-scattering in 2nd approximation:

1) The diagram for polarization of vacuum

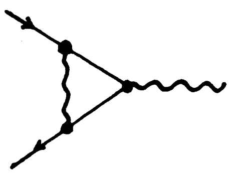


modifies the propagator of photon to:

$$iD_{\mu\nu}(q^2) = \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\alpha}}{q^2} \cdot I^{\alpha\beta}(q^2) \cdot \frac{-ig_{\beta\nu}}{q^2} \tag{1}$$

- The self energy of photon $I^{\alpha\beta}(q^2)$ diverges logarithmically.
- Problem of divergence we solved in such a way that the $I^{\alpha\beta}(q^2)$ we first regularized: $\int_0^\infty dp \cdots \;\rightarrow\; \int_0^M dp \cdots$ and then we renormalized charge, i.e. we made the replacement: $e \rightarrow e_R = e\left(1 - \frac{\alpha}{3\pi} \ln \frac{M^2}{m^2}\right)^{1/2}$ and e_R was declared as the charge observed in experiment.
- Working with e_R we have shown that effect of virtual pair $e^- e^+$ modifies potential between e^- and $Z\,e$ and the physical manifestation of these virtual pairs in hydrogen atom is Lamb shift.

The diagram for vertex correction

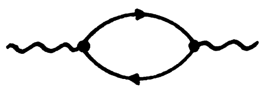


modifies the structure of electron current $(-e\bar{u}_f \gamma^\mu u_i)$: $\gamma^\mu \rightarrow \Gamma^\mu = \gamma^\mu + \Lambda^\mu$, this for small q^2 leads to:

$$-e\bar{u}_f \left\{ \underbrace{\gamma^\mu [1 + \cdots]}_{\text{modifikuje náboj}} - \underbrace{\left[\frac{\alpha}{3\pi} \frac{q^2}{m^2} \frac{i\sigma_{\mu\nu}}{2m} q^\nu \right]}_{\text{vedie k anomálnemu mag. momentu}} \right\} u_i \tag{2}$$

In general the 2nd order of perturbative expansion leads to the following divergent diagrams:

1) The self energy of photon (polarization of vacuum) setting $I^{\mu\nu} = -i\Pi^{\mu\nu}$:

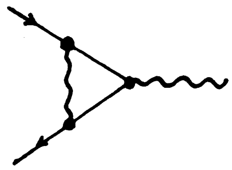


$$-i\Pi^{\mu\nu}(q^2) = -(ie)^2 \int \frac{d^4k}{(2\pi)^4} Tr\left[\gamma^\mu \cdot \frac{i(\hat{k}+m)}{k^2-m^2} \cdot \gamma^\nu \cdot \frac{i(\hat{k}-\hat{q}+m)}{(k-q)^2-m^2}\right] \tag{3}$$

2) The self energy of electron:

$$i\Sigma(p^2) = -(ie)^2 \int \frac{d^4k}{(2\pi)^4} \cdot \gamma^\mu \cdot \frac{i(\hat{p}-\hat{k}+m)}{(p-k)^2-m^2} \cdot \gamma^\nu \cdot \frac{-ig_{\mu\nu}}{k^2} \tag{4}$$

2) The vertex correction:



$$ie\Lambda^\mu(p,q,p+q) = (ie)^3 \int \frac{d^4k}{(2\pi)^4} \left(\frac{-ig_{\rho\sigma}}{(k+p)^2}\right)\left[\gamma^\rho \cdot \frac{i(\hat{p}-\hat{k}+m)}{(p-q)^2-m^2} \cdot \gamma^\mu \cdot \frac{i(\hat{k}+m)}{k^2-m^2}\gamma^\sigma\right] \tag{5}$$

The shown integrals contain divergences that are removed by the procedure of renormalization. We have shown it in the case of charge. The full removal of divergences in QED requires renormalization of also mass and wave function of particle.

Dimensional regularization

Up to now the renormalization was based on the so-call „cutoff" regularization. At present it is conventional to use the so-called ***dimensional regularization.*** In this regularization the divergent integrals are calculated in $D=4\pm 2\varepsilon$ ($D=4\pm\varepsilon$)-dimensions. We will show it on the case of the vacuum polarization, i.e. integral $I^{\mu\nu}=-i\Pi^{\mu\nu}$.

$$\Pi^{\mu\nu}=ie^2\left\{\int\frac{d^4k}{(2\pi)^4}\frac{Tr\left(\gamma^\mu(\hat{k}+m)\gamma^\nu(\hat{k}-\hat{q}+m)\right)}{\left(k^2-m^2\right)\left((k-q)^2-m^2\right)}\right\} \tag{6}$$

First a few words about the technique of calculation of the divergent integrals.

The technique of calculation of the divergent integrals

Calculation of traces of γ-matrices. For calculation of divergent diagrams of the type (6) is needed first of all to figure out traces γ-matrices. In the concrete for calculation of the quantity $\Pi_{\mu\nu}(q^2)$ it is needed to use the following properties of traces of γ-matrix product:

$$\begin{aligned}
Tr\left(\gamma^\mu\gamma^\nu\right) &= 4g^{\mu\nu}\\
Tr\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= 4\left[g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}-g^{\mu\nu}g^{\rho\sigma}\right]\\
Tr\left(\underbrace{\gamma^\mu\cdots\gamma^\sigma}_{odd}\right) &= 0
\end{aligned} \tag{7}$$

About the calculation of traces of matrices in more details can be read in the appendix A.

The technique of calculation of divergent integrals is based on a few tricks.

Feynman's trick. It is easily to show (see the appendix B – (B3)) that:

$$\frac{1}{\left(k^2-m^2\right)\left((k-q)^2-m^2\right)}=\int_0^1 dx\frac{1}{\left(k^2-2k\cdot qx+q^2x-m^2\right)^2}=\int_0^1 dx\frac{1}{\left(k'^2-M^2\right)^2} \tag{8}$$

where $M^2=x^2q^2-xq^2+m^2$ and $k'=k-qx$.

The transition to Euclidian integral in D dimensions. Typical expression with which we have to do at calculation of 2nd order of the perturbative expansion reads:

$$I\left(M^2,\alpha\right)=\int\frac{d^4k}{(2\pi)^4}\frac{1}{\left(k^2-M^2\right)^{\alpha}} \tag{9}$$

We carry out the regularization of the integral in such a way that we will do integration of the term under the integral in D dimensions. The function under the integral is an a analytic function with exception of poles ($\pm E=\pm\sqrt{\vec{k}^2+m^2}$) in variable k_0. The problem of the poles we have already solved (see Chapter 7) by moving to the complex plane of k_0. The poles we shifted: $\pm E\rightarrow\pm(E-i\varepsilon)$ and make the replacement: $\int_{-\infty}^{\infty}dk^0\cdots\xrightarrow{\pm E\rightarrow\pm(E-i\varepsilon)}\int_C dk^0\cdots$, where $C=(-\infty,\infty)\cup K$, $K\equiv$ is half-circumference in lower and upper half-plane of k_0, respectively.

As $\int_K dk^0\cdots=0\quad\Rightarrow\quad\int_{-\infty}^{\infty}dk^0\cdots=\lim_{\varepsilon\rightarrow 0}\int_C dk^0\cdots$ then due to analyticity of the function under the integral, the contour C can be deformed: $C\rightarrow C'\equiv(-i\infty,i\infty)\cup K'$, where K' is the half-circumference in the forward or backward half-plane of k_0. Using $\int_C dk^0\cdots=\int_{C'}dk^0\cdots$ one can write:

$$I_D\left(M^2,\alpha\right)=\int_{-i\infty}^{i\infty}\frac{dk^0}{2\pi}\int_{-\infty}^{\infty}\frac{dk^1}{2\pi}\cdots\int_{-\infty}^{\infty}\frac{dk^{D-1}}{2\pi}\frac{1}{\left(k^2-M^2\right)^{\alpha}}$$

This integral can be transformed to euclidian integral by replacement:

$$k^0=ik_E^0,\quad \vec{k}=\vec{k}_E,\quad d^Dk=id^Dk_E\quad\Rightarrow k^2=-k_E^2=-\left(\left(k_E^0\right)^2+\cdots+\left(k_E^{D-1}\right)^2\right),$$
$$(-i\infty,\ i\infty)\ \rightarrow\ (-\infty,\infty)$$

It gives:

$$I_D\left(M^2,\alpha\right)=\int\frac{d^Dk}{(2\pi)^D}\frac{1}{\left(k^2-M^2\right)^{\alpha}}=(-1)^{\alpha}i\int\frac{d^Dk_E}{(2\pi)^D}\frac{1}{\left(k_E^2+M^2\right)^{\alpha}} \tag{10}$$

The integral $I_D(M^2)$ is finite for $D\neq 4$. The idea of regularization is the following: to use the formal term $I_D(M^2)$ for $D\neq 4$ and carry out all needed manipulations – to regularize divergences, to renormalize fields and coupling constants and then go back to $D=4$.

Master formula. The calculation in (10) we can carry out as follows (we use the Euclidian metric – index „*E*“ we have omitted):

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{\left(k^2+M^2\right)} = \int \frac{d^D k}{(2\pi)^D} \int_0^\infty dz\, e^{-z\left(k^2+M^2\right)} = \int_0^\infty dz\, e^{-zM^2} \int \frac{d^D k}{(2\pi)^D} e^{-zk^2}$$
$$= \frac{1}{(2\pi)^D} \int_0^\infty dz\, e^{-zM^2} \left(\frac{\pi}{z}\right)^{D/2} = \frac{\left(M^2\right)^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \int_0^\infty dt\, e^{-t} t^{-\frac{D}{2}} = \frac{\Gamma(1-D/2)}{(4\pi)^{D/2}} \left(M^2\right)^{\frac{D}{2}-1} \tag{11}$$

If we derive the relation (11) by M^2 ($\alpha-1$)-times we get the so-called „master formula":

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{\left(k^2+M^2\right)^\alpha} = \frac{\Gamma(\alpha-D/2)}{(4\pi)^{D/2}\,\Gamma(\alpha)} \left(M^2\right)^{\frac{D}{2}-\alpha} \tag{12}$$

Remark. *In general it can be shown that on the basis of the integral* $I_D\left(M^2,\alpha\right)$ *we can calculate all integrals needed for calculation of the so-called n-point function – see appendix B.*

The expression (6) for $\Pi^{\mu\nu}$ can be adjusted by making the Feynman's trick and the replacement $k' = k - qx$

$$\Pi^{\mu\nu} = ie^2 \left\{ \int_0^1 dx \int \frac{d^4k'}{(2\pi)^4} \frac{Tr\left(\gamma^\mu(\hat{k}'+\hat{q}x+m)\gamma^\nu(\hat{k}'-\hat{q}(1-x)+m)\right)}{\left\{\left[(k'+qx)^2-m^2\right](1-x)+\left[k'-q(1-x)\right]x\right\}^2} \right\} \tag{13}$$

The terms proportional to k' do not give any contribution to the integral (13) because they are odd functions of the integration variable. The denominator of the term under integral can be adjusted in the following way:

$$\{\cdots\} = k'^2 + q^2 x(1-x) - m^2 \tag{14a}$$

And numerator in that integral is:

$$Tr[\cdots] = \left[\left(k'_\alpha + q_\alpha x\right)\left(k'_\beta - q_\beta(1-x)\right)\right] Tr\left(\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right) + m^2 Tr\left(\gamma^\mu\gamma^\nu\right) \Rightarrow$$
$$\Rightarrow \left[k'_\alpha k'_\beta - x(1-x) q_\alpha q_\beta\right] 4\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\alpha}g^{\nu\beta}\right) + m^2 4 g^{\mu\nu} = \tag{14b}$$
$$= 4\left[2k'^\mu k'^\nu - 2\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)x(1-x) - g^{\mu\nu}\left(k'^2 + q^2 x(1-x) - m^2\right)\right]$$

Where we have exploited the properties of traces of γ-matrices (7) and „⇒" means that we omitted from the expression the terms proportional to k' that do not give any contribution to the integral.

If we do the replacement $k' \to k$ and set $M^2 = m^2 - q^2 x(1-x)$, the expression (13) reads:

$$\Pi^{\mu\nu}=4ie^2\left\{\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\left[\frac{2k^\mu k^\nu}{\left(k^2-M^2\right)^2}-\frac{2\left(q^\mu q^\nu-g^{\mu\nu}q^2\right)x(1-x)}{\left(k^2-M^2\right)^2}-\frac{g^{\mu\nu}}{k^2-M^2}\right]\right\} \qquad (15)$$

The integral (15) we solve by going to *D*-dimensional space and by subsequent application of the „master formula“ and its modification (appendix B, B7-10) :

$$\int\frac{d^Dk}{(2\pi)^D}\frac{2k^\mu k^\nu}{\left(k^2-M^2\right)^2}=-ig^{\mu\nu}\frac{\Gamma(1-D/2)}{(4\pi)^{D/2}}\left(M^2\right)^{\frac{D}{2}-1}$$

$$\int\frac{d^Dk}{(2\pi)^D}\frac{1}{\left(k^2-M^2\right)^2}=-i\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}}\left(M^2\right)^{\frac{D}{2}-2} \qquad (16)$$

$$\int\frac{d^Dk}{(2\pi)^D}\frac{g^{\mu\nu}}{k^2-M^2}=-ig^{\mu\nu}\frac{\Gamma(1-D/2)}{(4\pi)^{D/2}}\left(M^2\right)^{\frac{D}{2}-1}$$

From (16) it follows that the first and third term in expression (15) are mutually cancelled and it leads to

$$\Pi^{\mu\nu}=8e^2\left(q^\mu q^\nu-g^{\mu\nu}q^2\right)\mu^{4-D}\frac{\Gamma\left(2-D/2\right)}{(4\pi)^{D/2}}\int_0^1 dx\, x(1-x)\left(M^2\right)^{\frac{D}{2}-2} \qquad (17)$$

At the transition to D-dimensional space we introduced an arbitrary scale μ that has the dimension of momentum and was introduced to make charge dimensionless also after the transition into D-dimensional space (see Appendix C). Now we set $D=4-\varepsilon$ and use:

$$\Gamma\left(2-D/2\right)=\Gamma\left(\frac{\varepsilon}{2}\right)\approx\frac{2}{\varepsilon}-\gamma+O(\varepsilon)$$

$$\left(M^2\right)^{-\frac{\varepsilon}{2}}\approx 1-\frac{\varepsilon}{2}\ln\left(M^2\right)=1-\frac{\varepsilon}{2}\ln\left(m^2-q^2x(1-x)\right) \qquad (18)$$

$$\frac{\mu^{4-D}}{(4\pi)^{\frac{D}{2}}}=\frac{\left(4\pi\mu^2\right)^{\frac{\varepsilon}{2}}}{(4\pi)^2}\approx\frac{1}{(4\pi)^2}\left(1+\frac{\varepsilon}{2}\ln\left(4\pi\mu^2\right)\right)$$

where $\gamma=0.577\ldots$ Euler's constant.

For the quantity $\Pi^{\mu\nu}$ we get:

$$\Pi^{\mu\nu} = \frac{e^2}{2\pi^2}\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)\left(\frac{2}{\varepsilon}-\gamma\right)\left(1+\frac{\varepsilon}{2}\ln\left(4\pi\mu^2\right)\right)\int_0^1 dx\, x(1-x)\left(1-\frac{\varepsilon}{2}\ln\left(m^2-q^2x(1-x)\right)\right)$$

$$= \frac{e^2}{2\pi^2}\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)\left(\frac{2}{\varepsilon}-\gamma+\ln\left(4\pi\mu^2\right)\right)\left(\frac{1}{6}-\frac{\varepsilon}{2}\int_0^1 dx\, x(1-x)\ln\left(m^2-q^2x(1-x)\right)\right)=$$

$$= \frac{e^2}{12\pi^2}\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)\left(\frac{2}{\varepsilon}-\gamma+\ln\left(4\pi\right)-\ln\frac{m^2}{\mu^2}-6\int_0^1 dx\, x(1-x)\ln\left(1-\frac{q^2}{m^2}x(1-x)\right)\right) \tag{19}$$

The term proportional to $q^\mu q^\nu$ does not give (after summing) any contribution to the process amplitude due to conservation of current ($j_\mu q^\mu=0$), therefore $\Pi^{\mu\nu}$ reads**:**

$$\Pi^{\mu\nu}\left(q^2\right) = -g^{\mu\nu}\Sigma^\gamma\left(q^2\right) = -g^{\mu\nu}q^2\Pi^\gamma\left(q^2\right) \tag{20}$$

where for the transversal self energy of photon we get:

$$\Sigma^\gamma = \frac{e^2}{12\pi^2}q^2\left(\frac{2}{\varepsilon}-\gamma+\ln\left(4\pi\right)-\ln\frac{m^2}{\mu^2}-6\int_0^1 dx\, x(1-x)\ln\left(1-\frac{q^2}{m^2}x(1-x)\right)\right) \tag{21}$$

Using the relations (1) and (20) for the photon propagator in 1-loop approximation we get:

$$D^{\mu\nu}(q^2) = -i\frac{g^{\mu\nu}}{q^2}\left[1-\Pi^\gamma(q^2)\right] \quad , \qquad \Pi^\gamma(q^2) = \frac{\Sigma^\gamma(q^2)}{q^2} \tag{22}$$

The case of small transferred momentum $|q^2| << m^2$**.** In this case one can write:

$\ln\left(1-\frac{q^2}{m^2}x(1-x)\right) \approx -\frac{q^2}{m^2}x(1-x)$ and for correction to propagator we get:

$$\Pi^\gamma \approx \frac{e^2}{12\pi^2}\left(\frac{2}{\varepsilon}-\gamma+\ln 4\pi-\ln\frac{m^2}{\mu^2}+6\frac{q^2}{m^2}\int_0^1 dx\, x^2(1-x)^2\right) \tag{23}$$

This leads to the following photon polarization of vacuum:

$$\Pi^\gamma(q^2) = \frac{\alpha}{3\pi}\left(\frac{2}{\varepsilon}-\gamma+\ln 4\pi-\ln\frac{m^2}{\mu^2}+\frac{q^2}{5m^2}\right) \tag{24}$$

The case of big transferred momentum $|q^2| >> m^2$**.**

$$\Pi^\gamma(q^2) = \frac{\alpha}{3\pi}\left(\frac{2}{\varepsilon}-\gamma+\ln 4\pi-\ln\frac{m^2}{\mu^2}+\ln\left(\frac{\left|q^2\right|}{m^2}\right)-\frac{5}{3}+i\pi\quad\theta(q^2)\right) \tag{25}$$

We used: $\ln\frac{x(1-x)q^2+m^2}{m^2}\approx\ln\frac{x(1-x)q^2}{m^2}=\ln x(1-x)+\ln\frac{|q^2|}{m^2}+i\pi\theta(q^2)$

Electron self energy

At the calculation of electron self energy we will go, similarly as in the case of polarization of vacuum, to ***D (= 4–ε)*** dimensional space:

$$-ie^2\int\frac{d^4k}{(2\pi)^4}\gamma^\mu\cdot\frac{i\left(\hat{p}-\hat{k}+m\right)}{(p-k)^2-m^2}\cdot\gamma^\nu\cdot\frac{-ig_{\mu\nu}}{k^2}\rightarrow\Sigma(p)=-ie^2\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\cdot\frac{\gamma^\mu\left(\hat{p}-\hat{k}+m\right)\gamma_\mu}{\left[(p-k)^2-m^2\right]k^2} \tag{26}$$

Having introduced the Feynmann variable ***z*** (see Appendix B2-3), we have:

$$\Sigma(p)=-ie^2\int_0^1 dz\,\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\cdot\frac{\gamma^\mu\left(\hat{p}-\hat{k}+m\right)\gamma_\mu}{\left[(p-k)^2z-m^2z+k^2(1-z)\right]^2} \tag{27}$$

After the replacement $k\rightarrow k'-pz$ and taking into account the fact that in ***D***-dimensional space is valid:

$$\gamma^\mu\gamma_\mu=D,\ \gamma_\mu\hat{a}\gamma^\mu=(2-D)\hat{a} \tag{28}$$

For the electron self energy we get:

$$\Sigma(p)=-i\frac{e^2}{(4\pi)^2}\left[\frac{1}{\varepsilon}\left(-\hat{p}+4m\right)+\hat{p}(1+\gamma)-2m(1+2\gamma)+2\int_0^1 dz\left(\hat{p}(1-z)-2m+\ln\left(\frac{M^2}{4\pi\mu^2}\right)\right)\right] \tag{29}$$

where $M^2=m^2z-p^2z(1-z)$

More details about the electron self energy can be found in Appendix C.

Vertex correction

In an analogical way as above it is possible to calculate the vertex correction:

$$-ie\mu^{2-D/2}\Lambda_\mu(p,q,p')=-\left(e\mu^{2-D/2}\right)^3\int\frac{d^Dk}{(2\pi)^D}\frac{\gamma_\nu\left(\hat{p}'-\hat{k}+m\right)\gamma_\mu\left(\hat{p}-\hat{k}+m\right)\gamma^\nu}{k^2\left[(p-k)^2-m^2\right]\left[(p'-k)^2-m^2\right]} \tag{30}$$

Using the 2-parametric Fenmann's formula (see Appendix B)

$$\frac{1}{abc}=2\int_0^1 dx\int_0^{1-x}dy\frac{1}{\left[a(1-x-y)+bx+cy\right]^3}$$

We get

$$\Lambda_\mu(p,q,p') = i2e^2\mu^{4-D}\int_0^1 dx \int_0^{1-x} dy \int \frac{d^D k}{(2\pi)^D} \frac{\gamma_\nu\left(\hat{p}'-\hat{k}+m\right)\gamma_\mu\left(\hat{p}-\hat{k}+m\right)\gamma^\nu}{\left[k^2-m^2(x+y)-2k(px+p'y)+p^2x+p'^2y\right]^3} \quad (31)$$

After introducing $k' = k - px - p'y$ and after renaming of integration variable ($k' \to k$) we have:

$$\Lambda_\mu(p,q,p') = i2e^2\mu^{4-D}\int_0^1 dx \int_0^{1-x} dy \int \frac{d^D k}{(2\pi)^D} \times$$
$$\times \frac{\gamma_\nu\left(\hat{p}'(1-y)-\hat{p}x-\hat{k}+m\right)\gamma_\mu\left(\hat{p}(1-x)-\hat{p}'y-\hat{k}+m\right)\gamma^\nu}{\left[k^2-m^2(x+y)+p^2x(1-x)+p'^2y(1-y)-2pp'xy\right]^3} \quad (32)$$

The expression (32) contains a finite as well as a divergent part (the part of numerator containing k^2 diverges therefore we can write:

$$\Lambda_\mu = \Lambda_\mu^{(1)} + \Lambda_\mu^{(2)} \quad (33)$$

where the divergent part ($\Lambda_\mu^{(1)}$) using (B10 – Appendix B), can be expressed in the form

$$\Lambda_\mu(p,q,p') = \frac{e^2}{2}\mu^{4-D}\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}}\int_0^1 dx \int_0^{1-x} dy$$
$$\times \frac{\gamma_\nu\gamma_\rho\gamma_\mu\gamma^\rho\gamma^\nu}{\left[m^2(x+y)-p^2x(1-x)-p'^2y(1-y)+2pp'xy\right]^{2-D/2}} \quad (34)$$

Now using the fact that

$$\gamma_\nu\gamma_\rho\gamma_\mu\gamma^\rho\gamma^\nu = (2-D)^2\gamma_\mu \quad (35)$$

and expressing $D=4-2\varepsilon$, the divergent part of the vertex correction reads:

$$\Lambda_\mu^{(1)}(p,q,p') = \frac{e^2}{(4\pi)^2\varepsilon}\gamma_\mu + \Lambda_\mu^{(1,\,finite)}(p,q,p') \quad (36)$$

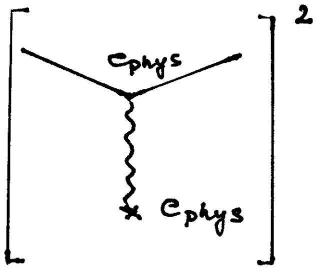
The convergent part, Λ_μ, does not contain k in the numerator and it converges so we can set $D=4$ and to integrate (over k). It will give

$$\Lambda_\mu^{(2)}(p,q,p') = \frac{e^2}{(4\pi)^2}\int_0^1 dx \int_0^{1-x} dy \frac{\gamma_\nu\left(\hat{p}'(1-y)-\hat{p}x+m\right)\gamma_\mu\left(\hat{p}(1-x)-\hat{p}'y+m\right)\gamma^\nu}{\left[m^2(x+y)-p^2x(1-x)-p'^2y(1-y)+2pp'xy\right]^{2-D/2}} \quad (37)$$

Conception of renormalization

From Ward Identity, that is a consequence of current conservation in QED, it follows that from the all one-loop diagrams (2nd order of perturbative expansion) only the polarization of vacuum contributes to the charge change (change of coupling constant in vertex). Let us assume that charge is measured in Thompson's scattering ($q^2 \approx 0$):

$\sigma_{thomp} \sim$



Then the amplitude in the process in the 2nd order of perturbative expansion we get from the amplitude of the 1st order:

$$ie\gamma^{\mu} \rightarrow ie\left[1 - \frac{1}{2}\Pi^{\gamma}(0)\right]\gamma^{\mu} \tag{38}$$

Hence in cross section for Thompson's scattering there is the quantity $e\left[1 - \frac{1}{2}\Pi^{\gamma}(0)\right]$ – it represents the physical charge (e_{phys}) and not the quantity that is in Lagrangian. Therefore the quantity that „sits“in Lagrangian we will denote as e_0 and will call it „bare“ charge. If the physical charge (e_{phys}), we will denote it as e, we measured at $q^2 \approx 0$, then the relation of bare (e_0) and physical charge (e) is:

$$e_0 = e + \delta e = e\left[1 + \frac{1}{2}\Pi^{\gamma}(0)\right] \Rightarrow \frac{\delta e}{e} = \frac{1}{2}\Pi^{\gamma}(0) \tag{39}$$

The recipe for the renormalization procedure is the following: physical charge in Lagrangian we replace by bare charge: $e \rightarrow e_0 = e + \delta e$ and then we carry out the expansion in δe.

Let us consider a scattering or an annihilation process that goes through photon exchange in one-loop approximation. The amplitude of scattering (without external fermion spinors) is:

$$\frac{e_0^2}{q^2}\left[1-\frac{1}{2}\Pi^\gamma(q^2)\right]^2=\frac{e^2}{q^2}\left[1+2\frac{\delta e}{e}-\Pi^\gamma(q^2)\right]=\frac{e^2}{q^2}\left[1+\Pi^\gamma(0)-\Pi^\gamma(q^2)\right]$$
$$=\frac{e^2}{q^2}\left[1-\hat{\Pi}^\gamma(q^2)\right] \tag{40}$$

The quantity

$$\hat{\Pi}^\gamma(q^2)=\Pi^\gamma(q^2)-\Pi^\gamma(0) \tag{41}$$

Is called the renormalized polarization of vacuum and is finite also for $D\to 4$. Using the relations (24) for small transferred momenta ($/q^2/ << m^2$):

$$\hat{\Pi}^\gamma(q^2)=\frac{\alpha}{3\pi}\frac{q^2}{5m^2} \tag{42}$$

For large transferred momenta (see Eq. 25) we get ($/q^2/ >> m^2$):

$$\hat{\Pi}^\gamma(q^2)=\frac{\alpha}{3\pi}\left(\frac{5}{3}-\ln\left(\frac{-q^2}{5m^2}\right)+i\pi\theta(q^2)\right) \tag{43}$$

At large $/q^2/$ to one-loop correction will contribute not only electrons but also other fermions with charge Q_f. An interesting case occurs when $-q^2=M_Z^2$ (M_Z is the mass of Z-boson):

$$\hat{\Pi}^\gamma(M_Z^2)=\sum_f Q_f^2\frac{\alpha}{3\pi}\left(\frac{5}{3}-\ln\left(\frac{M_Z^2}{5m^2}\right)+i\pi\theta(q^2)\right) \tag{44}$$

The total fermion contribution to the real part of the renormalized propagator is:

$$\mathrm{Re}\,\hat{\Pi}^\gamma(M_Z^2)=-0.0602\pm 0.0009$$

The polarization of vacuum can be treated either as a correction of propagator or as a correction of charge – in latter case a conception of running charge or running coupling constant. For the running charge we can sum the contributions of all orders (see Chapter 8 Eq. 37):

$$e^2(q^2)=\frac{e^2}{1+\mathrm{Re}\,\Pi^\gamma(q^2)} \tag{45}$$

It corresponds to summing of geometrical series representing a series of amplitudes with different number of loops.

Renormalization in QED

The conclusions for the divergent diagrams corresponding to the electron self energy, photon self energy and vertex correction after regularization are:

$$\Sigma(p) = \frac{e^2}{16\pi^2\varepsilon}(-\hat{p} + 4m) + \Sigma^{(\text{finit})}(p) \tag{46}$$

$$\Pi_{\mu\nu}(q) = \frac{e^2}{12\pi^2\varepsilon}(-g_{\mu\nu}q^2 + q_\mu q_\nu) + \Pi_{\mu\nu}^{(finit)}(p) \tag{47}$$

$$\Lambda_\mu(p,q,p') = \frac{e^2}{16\pi^2\varepsilon}\gamma_\mu + \Lambda_\mu^{(finit)}(p,q,p') \tag{48}$$

The diagrams (46-47) we have obtained starting from Lagrangian QED:

$$L_{QED} = \underbrace{\overline{\psi}(i\hat{\partial} - m)\psi}_{L_f} \underbrace{- eQA^\mu\overline{\psi}\gamma_\mu\psi}_{L_i} \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(\partial_\mu A^\mu\right)}_{L_\gamma} \tag{49}$$

Presence of the divergent parts in (46-48) means that the quantities ψ, m, e, A_μ present in L_{QED} are not the physically observable quantities but rather other quantities – we will call them „bare“ quantities and we will provide them with the suffix 0. Their relation to physically observable quantities can be expressed as follows:

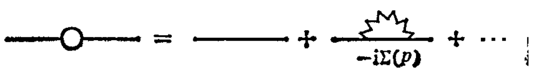
$$\begin{aligned} \psi_0 &= Z_\psi^{1/2}\psi \quad & A_0^\mu &= Z_A^{1/2}A_\mu \\ m_0 &= Z_m m \quad & e_0 &= Z_e e \end{aligned} \tag{50}$$

Where the quantities Z_i can be expanded into Taylor series in powers of square of the physical charge e^2:

$$Z_i = 1 + e^2\delta Z_i \tag{51}$$

Propagator electron

If we start from the Lagrangian L_{QED} (49) then the full electron propagator (dressed propagator) reads:



$$iS'_F(p) = \frac{i}{\hat{p}-m} + \frac{i}{\hat{p}-m}\left(-i\Sigma(p)\right)\frac{i}{\hat{p}-m} + \cdots = \frac{i}{\hat{p}-m-\Sigma(p)} \tag{52}$$

The effect of loop manifests itself as an addition to the mass of electron. The problem is that if in L_{QED} are physical quantities then due to presence of $\Sigma(p)$, propagator contains divergent part. Let us now suppose that into L_{QED} go the bare quantities, which we express through the physical ones using the relations (50) and (51). It is sufficient to take into account only the fermion part, L_f, of Lagrangian L_{QED}.

$$\left(L_f\right)_0 = \bar{\psi}_0(i\hat{\partial}-m)\psi_0 = (1+e^2\delta Z_\psi)\bar{\psi}\, i\hat{\partial}\psi - (1+e^2\frac{\delta m}{m} + e^2\delta Z_\psi)m\bar{\psi}\psi \tag{53}$$

The full propagator expressed through physical quantities can be obtained from (52) by

replacement: $\hat{p} \rightarrow \left(1+e^2\delta Z_\psi\right)\hat{p}$ a $m \rightarrow \left(1+e^2\frac{\delta m}{m}+e^2\delta Z_\psi\right)m$,

Hence the full propagator reads:

$$\begin{aligned} S'^{-1}_F &= \left(1+e^2\delta Z_\Psi\right)\hat{p} - \left(1+e^2\frac{\delta m}{m}+e^2\delta Z_\Psi\right)m - \Sigma = \\ &= \left(1+e^2\delta Z_\Psi + \frac{e^2}{8\pi^2\varepsilon}\right)\hat{p} - \left(1+e^2\frac{\delta m}{m}+e^2\delta Z_\Psi + \frac{e^2}{2\pi^2\varepsilon}\right)m - \Sigma^{(finit)} \end{aligned} \tag{54}$$

The expressions for the bare quantities we can get from the condition of finite value of expression (54):

$$\delta Z_\Psi = -\frac{1}{8\pi^2\varepsilon}, \qquad \frac{\delta m}{m} + \delta Z_\Psi + \frac{1}{2\pi^2\varepsilon} = 0 \;\;\Rightarrow\;\; \frac{\delta m}{m} = -\frac{3}{8\pi^2\varepsilon} \tag{55}$$

This leads to the following relation between the bare and physical quantities:

$$\Psi_0 = \sqrt{Z_2} = \left(1-\frac{e^2}{8\pi^2\varepsilon}\right)\Psi, \qquad m_0 = m + \delta m = \left(1-\frac{3e^2}{8\pi^2\varepsilon}\right) \tag{56}$$

And the full propagator reads:

$$iS'_F(p) = \frac{i}{\hat{p}-m-\Sigma^{(finit)}(p)} \tag{57}$$

Schemes of renormalization

In general for the vacuum polarization it can be written:

$$\Pi^{\gamma}(q^2) = \Delta\Pi_e^{\gamma}(\mu^2) + \Pi_R^{\gamma}(q^2/\mu^2) \tag{54}$$

where the divergent part $\Delta\Pi_e^{\gamma}(\mu^2)$ can be reabsorbed in renormalized coupling constant (charge). The decomposition (54) can be done in many different ways. The way which we take will defined renormalization scheme:

$$\Delta\Pi_e^{\gamma}(\mu^2) = \begin{cases} \frac{\alpha_0}{3\pi}\mu^{-2\varepsilon}\left[\frac{1}{\varepsilon} - \gamma + \ln(4\pi) + \frac{5}{3}\right] & \mu - \text{scheme} \\ \frac{\alpha_0}{3\pi}\mu^{-2\varepsilon}\left[\frac{1}{\varepsilon}\right] & \text{MS} - \text{scheme} \\ \frac{\alpha_0}{3\pi}\mu^{-2\varepsilon}\left[\frac{1}{\varepsilon} - \gamma + \ln(4\pi)\right] & \overline{\text{MS}} - \text{scheme} \end{cases} \tag{55}$$

$$\Pi_R(q^2/\mu^2) = \begin{cases} \frac{\alpha_0}{3\pi}\left[-\ln\left(\frac{-q^2}{\mu^2}\right)\right] & \mu - \text{scheme} \\ \frac{\alpha_0}{3\pi}\left[-\ln\left(\frac{-q^2}{\mu^2}\right) - \gamma + \ln(4\pi) + \frac{5}{3}\right] & \text{MS} - \text{scheme} \\ \frac{\alpha_0}{3\pi}\mu^{-2\varepsilon}\left[-\ln\left(\frac{-q^2}{\mu^2}\right) + \frac{5}{3}\right] & \overline{\text{MS}} - \text{scheme} \end{cases} \tag{56}$$

Using $\alpha = e^2/(4\pi)$ in QED we can write (the amplitude $M(q^2) \sim \frac{e^2}{q^2}\{1 - \Pi^{\gamma}(q^2)\}$):

$$\begin{aligned} &\frac{\alpha_0}{q^2}\{1 - \Delta\Pi_e^{\gamma}(\mu^2) - \Pi_R^{\gamma}(q^2/\mu^2)\} \equiv \frac{\alpha_R(\mu^2)}{q^2}\{1 - \Pi_R^{\gamma}(q^2/\mu^2)\}, \\ &\alpha_R(\mu^2) = \alpha_0\left\{1 + \frac{\alpha_0}{3\pi}\mu^{-2\varepsilon}\left(\frac{1}{\varepsilon} + C_{scheme}\right)\right\}, \qquad \alpha_0 = \frac{e_0^2}{4\pi} \end{aligned} \tag{57}$$

where similarly as in case of the „cut off“ renormalization, α_0 is the bare coupling constant, that is not directly observable. After redefinition coupling constant the scattering amplitude is finite, hence in experiment is measured the renormalized coupling constant α_R.

Appendix A. At calculation of vacuum polarization, electron self energy or vertex function it is needed to use the following properties of traces of γ-matrices product:

$$
\begin{aligned}
Tr\left(\gamma^{\mu}\gamma^{\nu}\right) &= 4g^{\mu\nu} \\
Tr\left(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right) &= 4\left[g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}-g^{\mu\nu}g^{\rho\sigma}\right] \\
Tr\left(\gamma_5\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right) &= 4i\varepsilon^{\mu\nu\rho\sigma} \\
Tr\left(\gamma_5\gamma^{\mu}\gamma^{\nu}\right) &= 0
\end{aligned}
\tag{A1}
$$

and the commutation relations ($\hat{a} \equiv a_{\mu}\gamma^{\mu}$) :

$$
\begin{aligned}
\hat{a}\hat{b} &= 2(ab)-\hat{b}\hat{a} \\
\hat{a}\gamma^{\mu} &= 2a^{\mu}-\gamma^{\mu}\hat{a} \\
\hat{a}\gamma^{5} &= -\gamma^{5}\hat{a}
\end{aligned}
\tag{A2}
$$

Appendix B: On calculation of divergent diagrams

The integral corresponding a Feynmann's diagram has the structure:

$$
I = \int \frac{f(k)dk}{a_1(k)a_2(k)\cdots a_n(k)} \tag{B1}
$$

where $a_i(k)$ are polynomials of second degree and $f(k)$ is a polynomial of n^{th} degree.

At the calculation is used:

$$
\frac{1}{a_1a_2\cdots a_n} = \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_0^{z_{n-1}} dz_{n-2} \frac{1}{\left[a_1z_{n-1}+a_2(z_{n-1}-z_{n-2})+\cdots+a_n(1-z_1)\right]^n} \tag{B2}
$$

For n=2:

$$
\frac{1}{a_1a_2} = \int_0^1 \frac{dz}{\left[a_1z+a_2(1-z)\right]} \tag{B3}
$$

From the view point of the dependence on k (momentum of integration) it is needed to make the replacement:

$$
\frac{1}{\left[a_1z_{n-1}+a_2(z_{n-1}-z_{n-2})+\cdots+a_n(1-z_1)\right]^n} = \frac{c}{\left[(k-a)^2+\alpha\right]^n} \tag{B4}
$$

where c, a, α are functions of $z_1,\ldots,z_n$.

Putting (B2) into (B1) we get:

$$
I = (n-1)!\int_0^1 dz_1 \cdots \int_0^{z_{n-2}} dz_{n-1} J(z_1,\ldots,z_{n-1}) \tag{B5}
$$

where

$$J(z_1,\cdots z_{n-1}) = \int dk \frac{f(k)}{\left[(k-a)^2+b\right]^n} \tag{B6}$$

For calculation of higher orders it is sufficient to know the integral (dimensional regularization):

$$J_\alpha\left(M^2\right) = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\left(k^2-M^2\right)^\alpha} = (-1)^\alpha i \frac{\Gamma(\alpha - D/2)}{(4\pi)^{D/2}\,\Gamma(\alpha)} \left(M^2\right)^{\frac{D}{2}-\alpha} \tag{B7}$$

The general case of scalar integral (see further) can be transformed by the replacement $k \to k+p$ to the case $J_\alpha(M^2)$:

$$J_\alpha\left(p,M^2\right) = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\left(k^2+2kp-M^2\right)^\alpha} = (-1)^\alpha i \frac{\Gamma(\alpha - D/2)}{(4\pi)^{D/2}\,\Gamma(\alpha)} \left(M^2+p^2\right)^{\frac{D}{2}-\alpha} \tag{B8}$$

Differentiating both sides of B8 in p^μ we get:

$$\int \frac{d^Dk}{(2\pi)^D} \frac{k_\mu}{\left(k^2+2kp-M^2\right)^\alpha} = (-1)^\alpha i\, p_\mu \frac{\Gamma(\alpha - D/2)}{(4\pi)^{D/2}\,\Gamma(\alpha)} \left(M^2+p^2\right)^{\frac{D}{2}-\alpha} \tag{B9}$$

Differentiating both sides of (B9) in p^ν we get:

$$\int \frac{d^Dk}{(2\pi)^D} \frac{k_\mu k_\nu}{\left(k^2+2kp-M^2\right)^\alpha} =$$

$$= (-1)^\alpha i \frac{\left(M^2+p^2\right)^{\frac{D}{2}-\alpha}}{(4\pi)^{D/2}\,\Gamma(\alpha)} \left[p_\mu p_\nu \Gamma\left(\alpha - \frac{D}{2}\right) + \frac{1}{2} g^{\mu\nu} \left(p^2+M^2\right) \Gamma\left(\alpha - 1 - \frac{D}{2}\right) \right] \tag{B10}$$

$$\xrightarrow[p=0]{} \; (-1)^\alpha \frac{i}{2} g^{\mu\nu} \frac{\Gamma\left(\alpha - 1 - \frac{D}{2}\right)}{(4\pi)^{D/2}\,\Gamma(\alpha)} \left(M^2\right)^{\frac{D}{2}-\alpha+1}$$

Appendix C: Dimensional analysis

Let us consider a physical system in D-dimensional space. Let us assume that Λ is a momentum unit, then the action of the system should be:

$$\left[\int d^Dx \cdot L(x)\right] = \Lambda^0 \quad \Rightarrow \quad [L] = \Lambda^D \quad \left(\left[d^Dx\right] = \Lambda^{-D}\right) \tag{C1}$$

Scalar field:

$$\left[\partial_\mu \varphi \partial^\mu \varphi\right] = \Lambda^D \quad \Rightarrow \quad [\varphi] = \Lambda^{\frac{D-2}{2}} \tag{C2}$$

Spinor field:

$$\left[\bar{\psi}\not{\partial}\psi\right]=\Lambda^{D} \quad \Rightarrow \quad \left[\psi\right]=\Lambda^{\frac{D-1}{2}} \qquad \text{C3}$$

Electromagnetic field:

$$\left[F^{\mu\nu}\tilde{F}_{\mu\nu}\right]=\left[\partial_{\mu}A_{\nu}\partial^{\mu}A^{\nu}\right]=\Lambda^{D} \quad \Rightarrow \quad \left[A\right]=\Lambda^{\frac{D-2}{2}} \qquad \text{C4}$$

Interaction term:

$$\left[e\bar{\psi}\not{A}\psi\right]=\Lambda^{D} \quad \Rightarrow \quad \left[e\right]=\Lambda^{\frac{4-D}{2}} \qquad \text{C5}$$

If we assume $D=4-\varepsilon$ and in the interaction term we do the replacement: $e\rightarrow e\mu^{\frac{4-D}{2}}=e\mu^{\frac{\varepsilon}{2}}$, then the charge e will stay dimensionless.