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Basic principles of QCD - the role of colored charge

Up to now we have assumed that quarks play the role of partons in proton. However in reality almost 50% of momentum of proton is carried by gluons. Therefore we will assume that partons are quarks and gluons. To take into account the role of gluons we need to go beyond the frame of the „naive“ parton model and take into account colored charge of partons.

Theory that deals with dynamics of colored charge is **quantum chromodynamics** (QCD). The basic principles which QCD is relied on are:

- Quarks have not only the electric charge but also the colored one (***R,G,B***).
- Exchange of colored charge is carried out by means of gluons.
- The „colored” interactions are analogical to the electromagnetic ones: Feynmann’s diagrams we get by the replacement $\sqrt{\alpha} \to \sqrt{\alpha_s}$ in each vertex, hence the structure ***qqg*** is formally equivalent to $ee\gamma$.
- Gluons carry colored charge, hence they can interact between them.
- At small distances (big Q^2) the coupling constant of strong interaction α_S is small – hence the perturbative theory is applicable.

We will show how will be change the picture of the ***ep***-scattering, when we take into account the dynamics of colored charge. The basic manifestation of the gluon dynamics are:

- Quark can radiate gluon before and after interaction with virtual photon.
- Target gluon can give a contribution through production of pairs: $\gamma^* g \to q\bar{q}$.

The basic manifestation of the gluon dynamics we get if into the ***ep***-scattering we include process of order $\alpha\alpha_S$ (see *Fig.1*), and not only electro-magnetic processes (processes of order α).

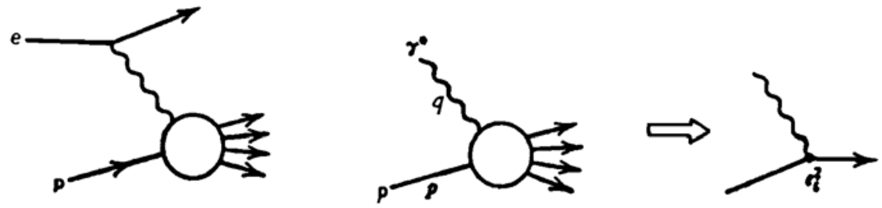
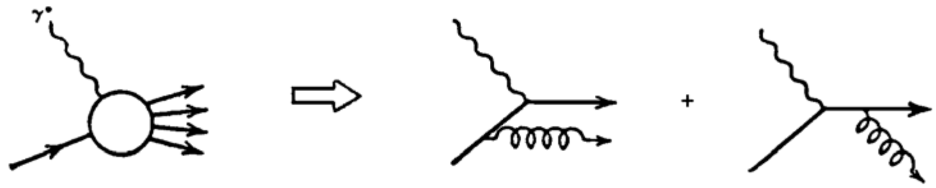
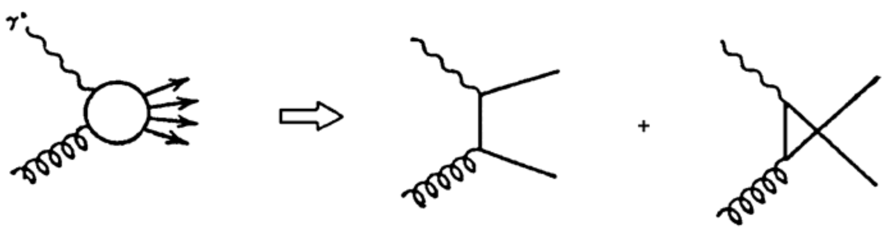


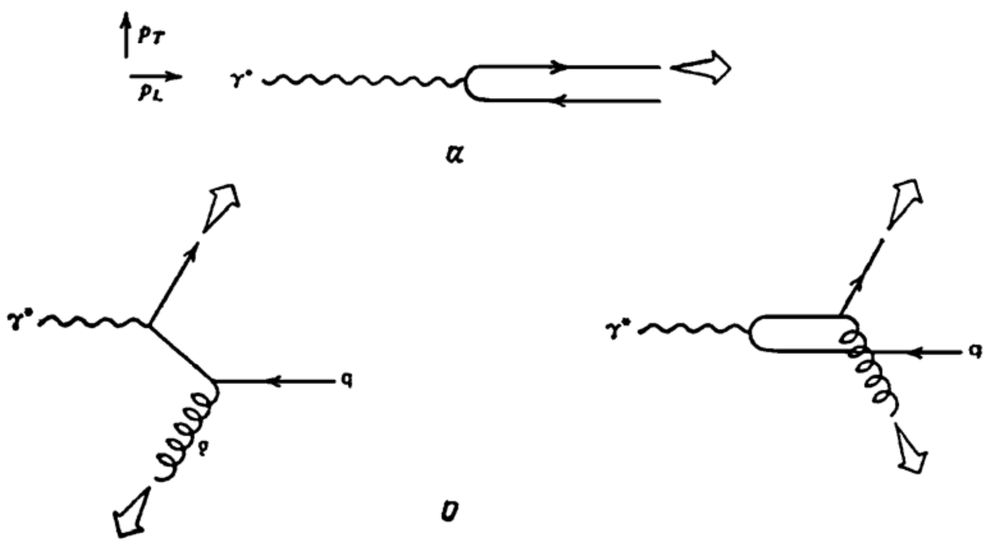
Diagram partónového modelu pre proces $ep \rightarrow eX$



Vklad rádu $\alpha\alpha_s$ $(\gamma^* q \rightarrow qg)$ do účinného prierezu procesu $ep \rightarrow eX$.



Vklad (rádu $\alpha\alpha_s$) iniciovaný gluónom hlboko–nepružnej zrážky $(\gamma^* g \rightarrow q\bar{q})$ do účinného prierezu procesu $ep \rightarrow eX$.



a) diagram partónového modelu pre proces $\gamma^* q \rightarrow q$ vudúceho k zrodu jetu s $p_T=0$
b) diagramy s vyžiarením gluónu zodpovedajuceho vzniku jetu s $p_T \neq 0$

Fig. 1: Including of processes of the order $\alpha\alpha_S$ into the *ep*-scattering.

Consequences of inclusion of the processes of order $\alpha\alpha_S$

The main consequences of inclusion of the processes with gluons are:

- Scaling violation for structure functions.
- Quark of final state will non-collinear with virtual photon, hence his $p_T \neq 0$ (with respect to direction of virtual photon).

The $ep \to eX$ cross section is given by the inelastic formfactors:

$$F_1 = MW_1\left(\nu, Q^2\right) \text{ a } F_2 = \nu W_2\left(\nu, Q^2\right), \tag{1}$$

That after inclusion of processes of order $\alpha\alpha_S$ into the *ep*-scattering the formfactors will be the functions of ν $\left(= p\cdot q/M\right)$ and Q^2 $\left(=-q^2\right)$, what is a consequence of the scaling violation and not only the function $\omega = 2M\nu/Q^2$.

For the relation between the proton and parton cross section ($\gamma^* p$ vs γ^* *parton*) is important to realize that in deep-inelastic limit is valid:

$$2F_1 = \frac{\sigma_T}{\sigma_0} \text{ a } \frac{F_2}{x} = \frac{\sigma_T + \sigma_L}{\sigma_0} \tag{2}$$

where σ_T, σ_L are cross section of absorption of transverse and longitudinal photons and $\sigma_0 = 4\pi^2\alpha/s$. The relations (2) refer to the γ^*-proton and not to the γ^*-parton cross section. To go to the parton level, let us consider the process of gluon radiation $\gamma^* q \to qg$. A comparison of the proton and parton scattering is in the following table:

γ^***protón**	$\to$	γ^***partón**
p	$\to$	$p_i = yp$
$x = \dfrac{Q^2}{2p\cdot q}$	$\to$	$z = \dfrac{Q^2}{2p_i\cdot q} = \dfrac{x}{y}$

The relation between σ_T/σ_0 (proton scattering) and $\hat{\sigma}_T/\hat{\sigma}_0$ (parton scattering) is:

$$\left(\frac{\sigma_T\left(x,Q^2\right)}{\sigma_0}\right)_{\gamma^* p} = \sum_i \int_0^1 dz \int_0^1 dy\, f_i(y)\,\delta(x - zy)\left(\frac{\hat{\sigma}_T\left(z,Q^2\right)}{\hat{\sigma}_0}\right)_{\gamma^* i} \tag{3}$$

where $f_i(y)$ is the structure function of parton "*i*" in proton, i.e. the probability density of the hit parton to have relative momentum y,

$\hat{\sigma}_T\left(z,Q^2\right)$ is the cross section of absorption of transverse photon by parton with relative momentum y including also the processes of the order $\alpha\alpha_S$,

$\delta(x-zy)$ provides that the variable x (characterizes $\gamma^* p$-process) will be fixed: parton after interaction (absorption of γ^* and radiation of gluon) will have the momentum $p_{out} = \underbrace{xp}_{\gamma^* proton} = \underbrace{zp_i}_{\gamma^* parton}$. After the integration through z we get:

$$\left(\frac{\sigma_T\left(x,Q^2\right)}{\sigma_0}\right)_{\gamma^* p} = \sum_i \int_0^1 \frac{dy}{y} f_i(y) \left(\frac{\hat{\sigma}_T\left(\frac{x}{y},Q^2\right)}{\hat{\sigma}_0}\right)_{\gamma^* i} \tag{4}$$

Gluon emission

The QCD-process $\gamma^* q \to qg$ is from the view point of „diagram structure“ analogical to the QED-process $\gamma^* e \to \gamma e$. For example diagram in Fig. 2 has a similar structure as the diagram for Compton's effect (Fig. 3):

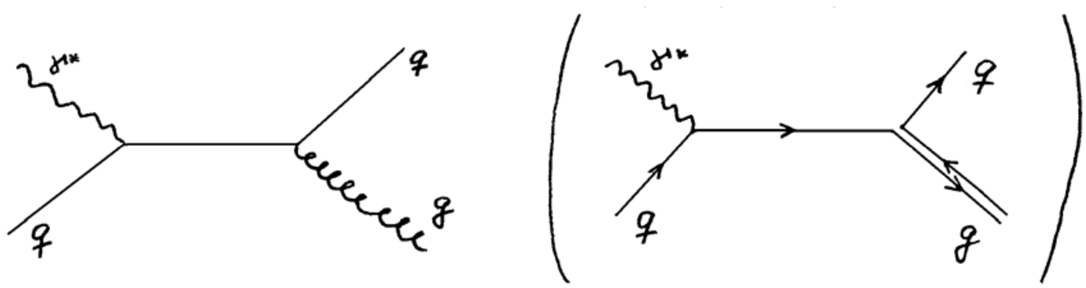
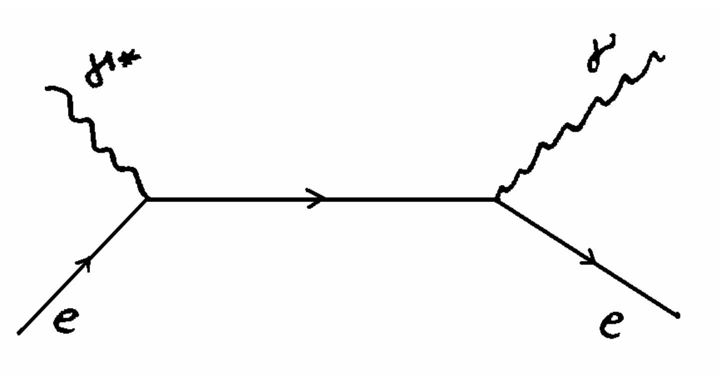


Fig. 2: The diagram of the process $\gamma^* q \to qg$ and diagram showing the flow of colored charge.



The square of module of amplitude summed through final spins and averaged through initial spins reads (5):

$$\overline{|M|^2} = 32\pi\alpha^2\left(-\frac{u}{s}-\frac{s}{u}+\frac{2tQ^2}{su}\right)$$

Fig. 3: The diagram Compton's scattering – the amplitude expressed through Mandelstam's variables (Appendix A).

The amplitude for the process $\gamma^* q \to qg$ we get from the amplitude of the process $\gamma^* e \to \gamma e$ by the replacement:

- $\alpha^2 \to e_i^2 \alpha \alpha_S$,
- We introduce the „colored“ coefficient 4/3,
- We carry out the replacement $u \leftrightarrow t$ that is connected with different order of emerge of final particles (if compared with the eγ^*-process) $\Rightarrow$ line of original quark goes into final gluon, while in the $\gamma^* e \to \gamma e$ it goes into final electron.

Therefore the amplitude of the process $\gamma^* q \to qg$ can be expressed as

$$\overline{|M|^2} = 32\pi e_i^2 \alpha \alpha_S \frac{4}{3}\left(-\frac{\hat{t}}{\hat{s}} - \frac{\hat{s}}{\hat{t}} + \frac{2\hat{u}Q^2}{\hat{s}\hat{t}}\right), \tag{6}$$

where $\hat{s}$, $\hat{t}$ a $\hat{u}$ are related to partons.

Why the coefficient of 4/3?

Each colored line can carry 3 colors (***R, G, B***) . at the same time radiated gluon can be one of the 8 different bi-colored charge combinations (the color singlet does not carry colored charge). The averaged through 3 colored initial state gives 8/3. The factor 2 if compared with Eq. (6) is a consequence of „historical“ definition of α_S.

Now we will be interested in the transverse momentum of quark (interacting with virtual photon) with respect of direction of virtual photon $p_T = k' \cdot \cos\theta$ (see Fig. 4).

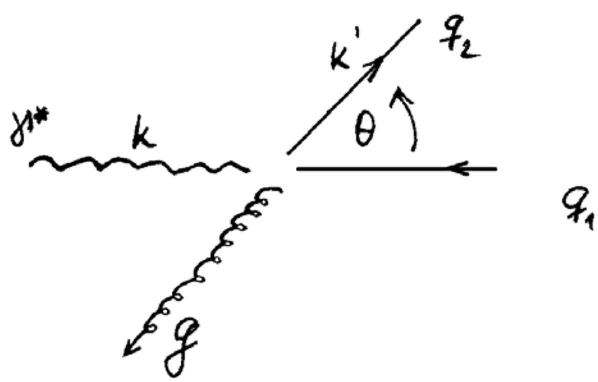


Fig. 4: **radiation of gluon by quark.**

It is valid:

$$\hat{s} = 2k^2 + 2k \cdot q_0 - Q^2 = 4k'^2$$
$$\hat{t} = -2k \cdot k'(1 - \cos\theta) \tag{7}$$
$$\hat{u} = -2k \cdot k'(1 + \cos\theta)$$

$\hat{s}, \hat{t}, \hat{u}$ are Mandelstam’s variables on parton level.

From the relation (7) it follows

$$p_T^2 = \frac{\hat{s}\hat{t}\hat{u}}{\left(\hat{s}+Q^2\right)^2} \tag{8}$$

At small scattering angle ($-\hat{t} \ll \hat{s}$) we can write:

$$p_T^2 = -\frac{\hat{s}\hat{t}}{\hat{s}+Q^2}, \qquad d\Omega = \frac{4\pi}{\hat{s}} dp_T^2 \tag{9}$$

If we use the relation between cross section and $\overline{|M|^2}$, for small angle we get:

$$\frac{d\sigma}{dp_T^2} \approx \frac{1}{16\pi\hat{s}^2}\overline{|M|^2} = -\frac{8\pi e_i^2}{3\hat{s}^2}\alpha\alpha_S \frac{1}{\hat{t}}\left(\hat{s} + \frac{2\left(\hat{s}+Q^2\right)Q^2}{\hat{s}}\right) \tag{10}$$

Introducing the variable $z = \frac{Q^2}{2p_i \cdot q} = \frac{Q^2}{\hat{s}+Q^2}$ we have:

$$\frac{d\sigma}{dp_T^2} \approx -e_i^2\hat{\sigma}_0 \frac{1}{p_T^2}\frac{\alpha_S}{2\pi}P_{qq}(z) \tag{11}$$

where

$$\hat{\sigma}_0 = \frac{4\pi^2\alpha}{\hat{s}} \quad \text{a} \quad P_{qq}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right) \tag{12}$$

P_{qq} is probability density of quark to radiate gluon and thereby to decrease its momentum z-times ($\tilde{p} \to z\tilde{p}$).

The singularity at $z \to 1$ is connected with radiation of soft gluon (infrared catastrophe). It is possible to show that the singularity will disappear if we take into account also diagrams with virtual gluons.

The process $\gamma^* q \to qg$ gives the main contribution for $-\hat{t} \ll \hat{s}$. In other cases it is needed to take into account also the process of creation of $q\bar{q}$ pairs (see further).

Experiment univocally confirms that radiation of gluon leads to existence of quark and gluon jet in final state and their directions does not correspond to that of virtual photon. An example of distribution of hadrons, in p_T^2, coming from the interaction μN is in Fig. 5.

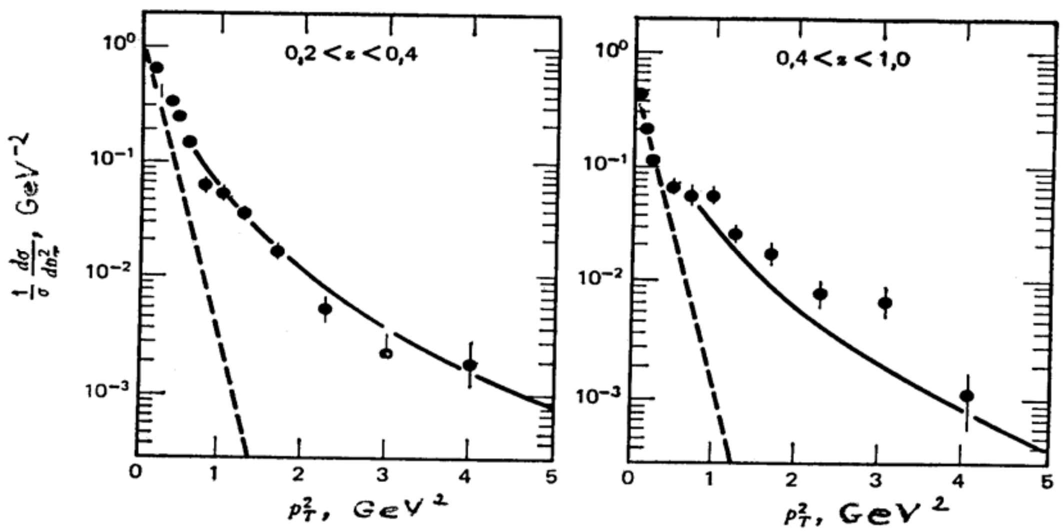


Fig. 5: Distribution of hadrons coming from the interaction μN in p_T^2.

In parton model without gluons are all jets collinear with virtual photon (slashed line in Fig. 5) – a small transverse momentum is a consequence of coupling of the interacting quark to other quarks.

Altarelli – Parisi (DGLAP) equation

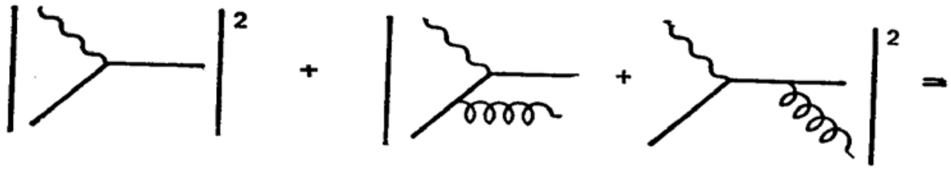
The cross section of radiation of bremstrahlung gluon is (see the relation(11)):

$$\hat{\sigma}\left(\gamma^* q \to qg\right) = \int_{\mu^2}^{\hat{s}/4} dp_T^2 \frac{d\hat{\sigma}}{dp_T^2} \approx e_i^2 \hat{\sigma}_0 \int_{\mu^2}^{\hat{s}/4} \frac{dp_T^2}{p_T^2} \frac{\alpha_S}{2\pi} P_{qq}(z) \approx \approx e_i^2 \hat{\sigma}_0 \frac{\alpha_S}{2\pi} P_{qq}(z) \ln \frac{Q^2}{\mu^2} \tag{13}$$

where we used the fact that $\left(p_T^2\right)_{\max} = \frac{\hat{s}}{4} = Q^2 \frac{1-z}{4z}$ and that at large Q^2 is valid: $\ln \frac{\hat{s}}{4} \approx \ln Q^2$. The integration from μ^2 means that we do not takeinto account the soft gluons.

The cross section of $\gamma^* q$-interaction including also radiation of gluon reads:

$$\frac{F_2\left(x,Q^2\right)}{x} =$$



$$= \sum_q e_q^2 \int_x^1 \frac{dy}{y} q(y) \left(\delta\left(1-\frac{x}{y}\right) + \frac{\alpha_S}{2\pi} P_{qq}\left(\frac{x}{y}\right) \ln\frac{Q^2}{\mu^2} \right) \tag{14}$$

The proton structure function F_2 is not only a function of variable x, but also of Q^2. The dependence on Q^2 is only logarithmic however it leads to violation of the Bjorken scaling. This violation is a result of gluon radiation. The fact that the structure function F_2 is not only a function of x, but also Q^2 is well demonstrated in Fig.6.

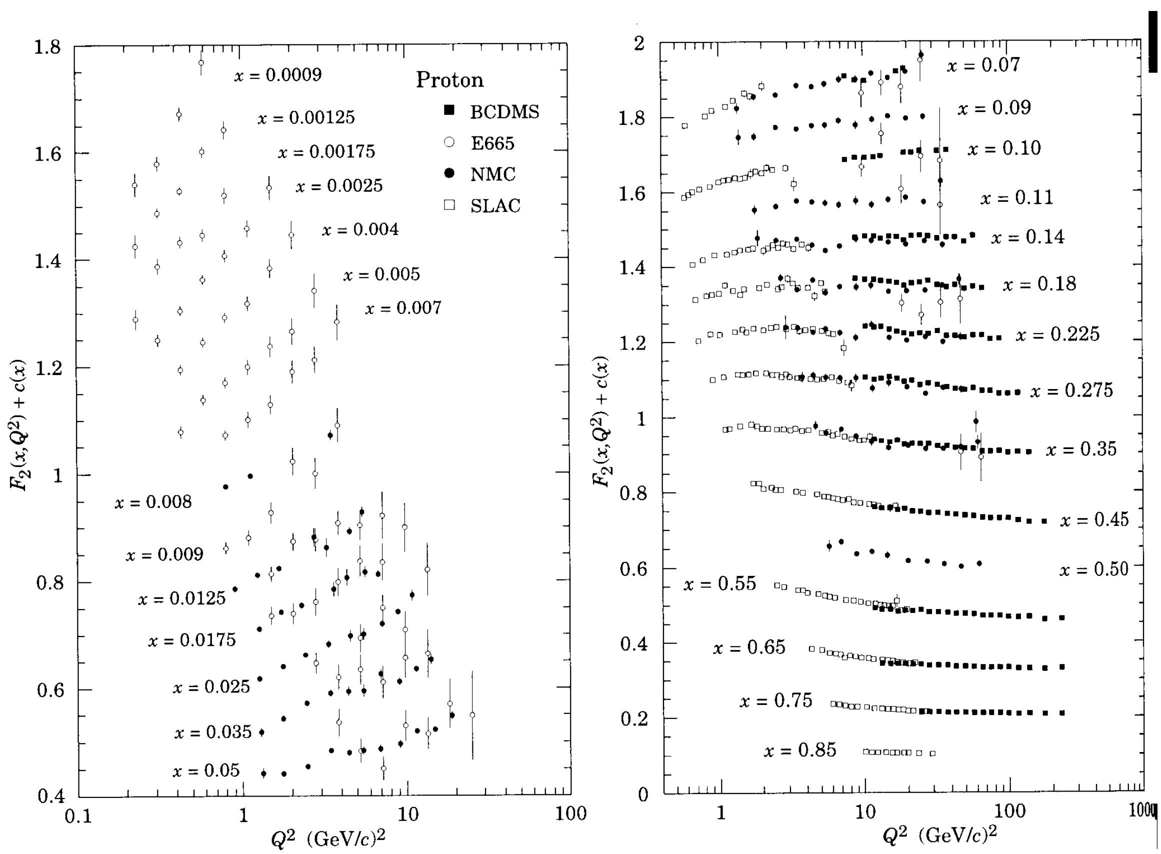


Fig. 6: The proton's structure function F_2 as a function of Q^2 for different x intervals.

Evolution of quark densities

The equation for $F_2\left(x,Q^2\right)/x$ we can write as follows:

$$\frac{F_2\left(x,Q^2\right)}{x}=\sum_q e_q^2\int_x^1\frac{dy}{y}\left(q(y)+\Delta q\left(y,Q^2\right)\right)\delta\left(1-\frac{x}{y}\right) \quad = \quad \sum_q e_q^2\left[q(x)+\Delta q\left(x,Q^2\right)\right]$$

(15)

where

$$\Delta q\left(x,Q^2\right)=\frac{\alpha_S}{2\pi}\ln\frac{Q^2}{\mu^2}\int_x^1\frac{dy}{y}q(y)P_{qq}\left(\frac{x}{y}\right) \tag{16}$$

Hence the quark density $q(x,Q^2)$ depends on Q^2 (see *Fig. 7*).

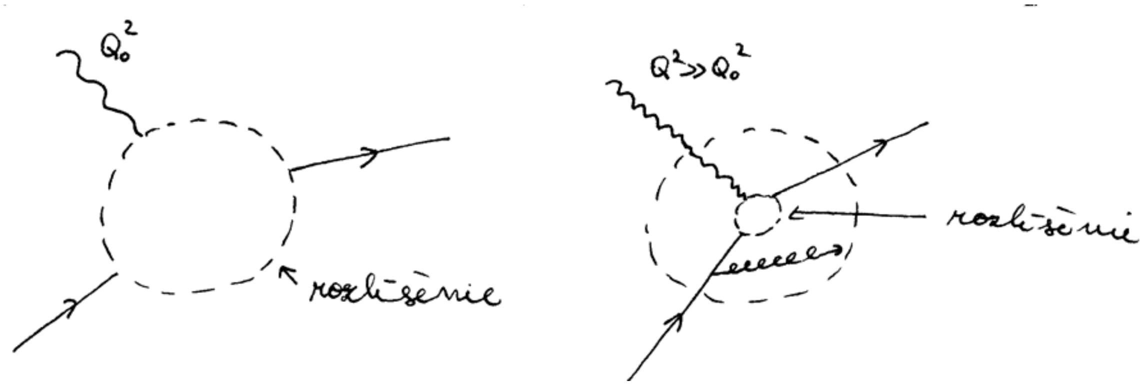


Fig. 7: Dependence of quark density on Q^2 of virtual photon.

At a certain Q_0^2 photon starts „to see" parton structure of proton. If quarks would not interact the picture of proton structure would not be changed. According to the QCD due to the strong interaction each quark is surrounded by cloud of partons – this fact must be "seen" by photon at $Q^2 \gg Q_0^2$. If we express the change of quark density Δq from the interval $\Delta \ln Q^2$ we get:

$$\frac{dq\left(x,Q^2\right)}{d\ln Q^2}=\frac{\alpha_S}{2\pi}\int_x^1\frac{dy}{y}q(y,Q^2)P_{qq}\left(\frac{x}{y}\right) \tag{17}$$

It is the evolution Altarelli–Parisi equation (now usually Dokhshitzer-Gribov-Lipatov-Altarelli-Parisi, DGLAP). Its interpretation is the following: quark with a relative momentum $\boldsymbol{x}$ (left side of equation) can arise from a quark with a bigger relative momentum ($\boldsymbol{y}$) that has radiated gluon.

The significance of DGLAP is in the following: if we know the quark structure function at a certain $Q^2 = Q_0^2$ we can calculate it at any other Q^2 using DGLAP.

Production of pairs by gluon

Up to now we have taken into account the process $\gamma^* q \to qg$ that gives the main contribution for $-\hat{t} \ll \hat{s}$. In other cases it is needed to take into account also the process of $q\bar{q}$ pair production by gluon. Gluon of proton can interact with virtual photon by means of the process $\gamma^* g \to q\bar{q}$ (see Fig. 8).

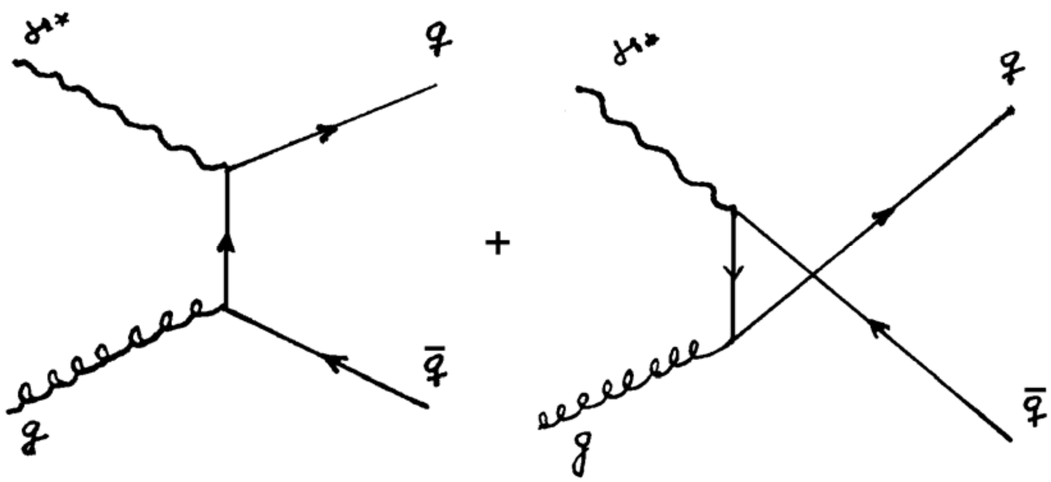


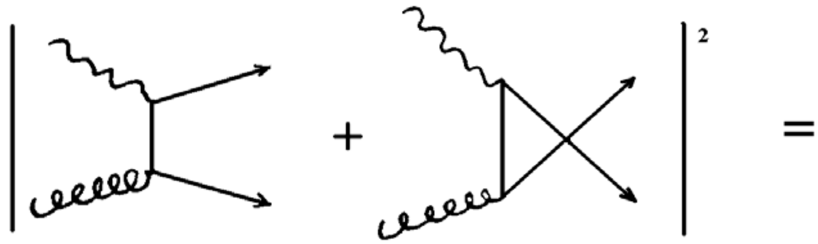
Fig. 8: Production of $q\bar{q}$ -pairs at interaction of virtual photon with gluon of proton.

The amplitude of process we get from the amplitude of Compton scattering by the replacement $\hat{s} \leftrightarrow -\hat{t}$.

$$\overline{|M|^2} = 32\pi^2 \left(e_q^2 \alpha\alpha_S\right) \frac{1}{2} \left(\frac{\hat{u}}{\hat{t}} + \frac{\hat{t}}{\hat{u}} - \frac{2\hat{s}Q^2}{\hat{t}\,\hat{u}} \right) \tag{18}$$

The contribution of the mentioned process to the structure function of proton is:

$$\left.\frac{F_2(x,Q^2)}{x}\right|_{\gamma^* g\to q\bar q} =$$



$$= \sum_q e_q^2 \int_x^1 \frac{dy}{y} g(y) \frac{\alpha_S}{2\pi} P_{qg}\left(\frac{x}{y}\right) \ln\frac{Q^2}{\mu^2} \qquad (19)$$

Where ***g(y)*** is the density of gluon in proton and

$$P_{qg}(z) = \frac{1}{2}\left(z^2 + (1-z)^2\right), \qquad (20)$$

is a probability of gluon to annihilate to $q\bar q$ pair and at the same time the quark carries the fraction z of his momentum.

The process $\gamma^* q \to q\bar q$ modifies the DGLAP in the following way:

$$\frac{dq_i(x,Q^2)}{d\ln Q^2} = \frac{\alpha_S}{2\pi}\int_x^1 \frac{dy}{y}\left[q_i(y,Q^2)P_{qq}\left(\frac{x}{y}\right) + g(y,Q^2)P_{qg}\left(\frac{x}{y}\right)\right] \qquad (21)$$

where i is the type of quark, g-term take into account the probability of quark with the momentum fraction x can be a result of creation of $q\bar q$ pair by an initial gluon with a momentum fraction $y\ (>x)$ – this probability is $P_{qg}(x/y)$.

A similar DGLAP it is possible to write also for gluon density $g(x,Q^2)$:

$$\frac{dg(x,Q^2)}{d\ln Q^2} = \frac{\alpha_S}{2\pi}\int_x^1 \frac{dy}{y}\left[q_i(y,Q^2)P_{gq}\left(\frac{x}{y}\right) + g(y,Q^2)P_{gg}\left(\frac{x}{y}\right)\right] \qquad (22)$$

The sum through i is taken through all quarks and anti-quarks of all colors and

$$P_{gg}(z) = 6\left(\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right) \qquad (23)$$

If we neglect the masses of quarks then P_{gg} does not depend from index i.

Physical interpretation of the function P

Let us consider the process $ep \to eX$ at large Q^2 and let us assume a possibility of gluon radiation by interacting parton. Due to this process there occurs a change of quark density in proton. And in accordance with (15) we can write:

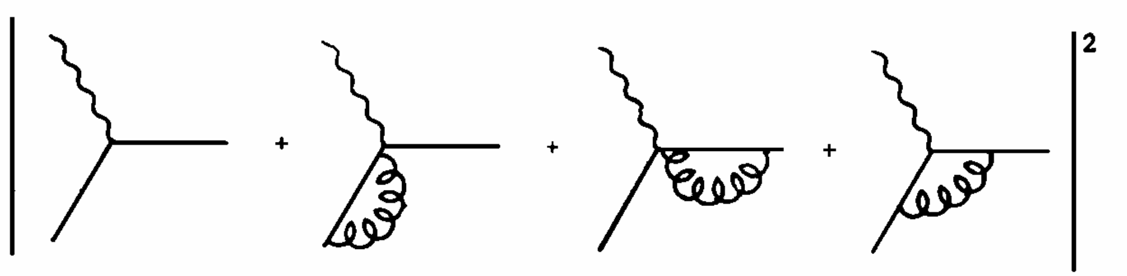
$$q(x)+\Delta q\left(x,Q^2\right)=\int_0^1 dy\int_0^1 dz\left[q(y,Q^2)\left(\delta(1-z)+\frac{\alpha_S}{2\pi}P_{qq}(z)\ln\frac{Q^2}{\mu^2}\right)\delta(x-zy)\right]=$$
$$=\int_0^1 dy\int_0^1 dz\left[q(y,Q^2)\Pi_{qq}(z,Q^2)\delta(x-zy)\right] \tag{24}$$

where $$\Pi_{qq}(z,Q^2)=\delta(1-z)+\frac{\alpha_S}{2\pi}P_{qq}(z)\ln\frac{Q^2}{\mu^2} \tag{25}$$

it is possible to interpret as a probability density to find in quark a quark carrying the fraction z of momentum of the original quark in the first order in α_S. The term $\delta(1-z)$ means that quark will stay without any change (does not radiate gluon). A problem of the relations (24, 25) is in the following:

1. there are not included the all needed diagrams giving the contribution $\sim \alpha_S$ – we take into account only the diagrams with radiation of real gluon.
2. The quantity $P_{qq}(z)=\frac{4}{3}\left(\frac{1+z^2}{1-z}\right)$ diverges for $z=1$ (infrared catastrophe).

The problem can be solved if we take into account also the diagrams with virtual gluons, concretely it means the first term (14) we replace by an extended term:



This term gives a contribution of order $\alpha\alpha_S$ that corresponds to interference of the first diagram with the diagrams containing virtual photons. The contributions of the interference terms exhibit also a singularity at $z = 1$ and moreover this singularity is exactly canceled with singularity present in (25) – due to the so-called. Block–Nordsiek theorem (Appendix B). Practically, singularity of type ***1/(1-z)*** of function P_{qq} is regularized by introducing the so-called „plus“-distribution that is got by the replacement:

$$1/(1-z) \quad \to \quad 1/(1-z)_+$$

where $1/(1-z)_+$ is defined to be valid:

$$\int_0^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \frac{f(z)-f(1)}{(1-z)} \tag{26}$$

After introducing the virtual correction the quantity $P_{qq}(z)$ is modified as follows:

$$P_{qq}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)_+ + 2\delta(1-z) \tag{27}$$

where δ-function is a consequence of presence of virtual correction.

Appendix A. Mandestam's variables

Let us consider two-particle process (in input as well as in output channel there are two particles). The process kinematic variables can be alternatively presented by the so-called Mandelstam's variables:

$$\begin{aligned} s &= (k+p)^2 \approx 2k\cdot p \approx 2k'\cdot p' \\ t &= (k-k')^2 \approx -2k\cdot k' \approx -2p\cdot p' \\ u &= (k-p')^2 \approx -2k\cdot p' \approx -2k'\cdot p \end{aligned} \tag{A1}$$

where $k, p\,(k', p')$ are momenta of input (output) particles.

An advantage of the variables ***s, t*** a ***u*** is the fact that they are invariants of the Lorentz group.

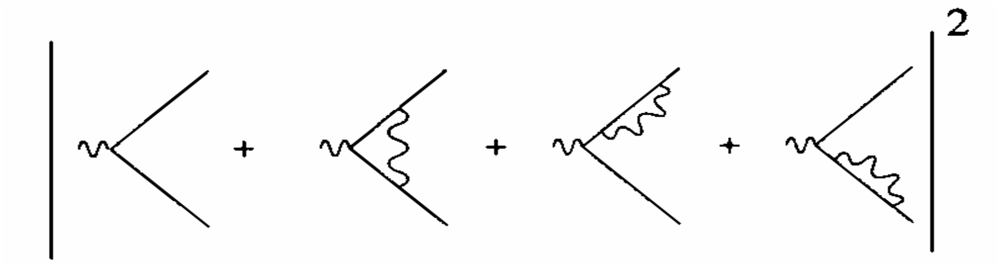
AppendixB. Block –Nordsiek theorem[1]

The infrared singularities of diagrams with real and virtual diagrams are mutually canceled. Let us consider the process $e^+e^- \to h$. In the first order of perturbative theory it is needed to take into account the processes $e^+e^- \to q\bar{q}$ and $e^+e^- \to q\bar{q}g$. Let us further assume that gluon has a small mass m_g, then for the cross sections of above mentioned processes we get:

[1] V.M. Braun, Application of QCD

The process $e^+e^- \rightarrow q\bar{q}$ with virtual corrections:

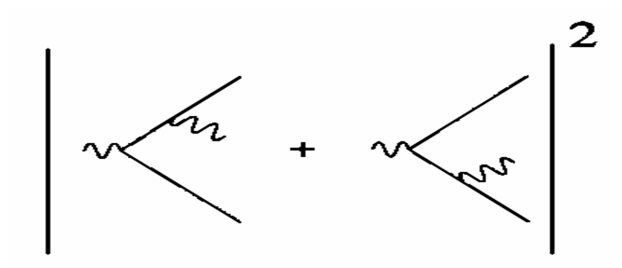
$$\sigma_{e^+e^-\rightarrow q\bar{q}} =$$



$$= \qquad \sigma_0 \left\{ 1 + \frac{4}{3}\frac{\alpha_S}{\pi}\left[-2\ln^2\frac{Q}{m_g} + 3\ln\frac{Q}{m_g} - \frac{7}{4} + \frac{\pi^2}{6} \right] \right\} \qquad \text{(B.1)}$$

The process $e^+e^- \rightarrow q\bar{q}g$ with a real gluon radiation:

$$\sigma_{e^+e^-\rightarrow q\bar{q}g} =$$



$$= \qquad \sigma_0 \frac{4}{3}\frac{\alpha_S}{\pi}\left[+2\ln^2\frac{Q}{m_g} - 3\ln\frac{Q}{m_g} + \frac{5}{2} - \frac{\pi^2}{6} \right] \qquad \text{(B.2)}$$

The cross section of both above mentioned are divergent for $m_g \rightarrow 0$. However in sum of these cross sections the singularities are mutually cancelled and this sum is independent from m_g and is finite:

$$\sigma_{tot}^{e^+e^-} = \sigma_0\left[1 + \frac{\alpha_S}{\pi} + O\left(\alpha_S^2\right) \right] \qquad \text{(B.3)}$$